EM Waves in Conductors

- So far we have solved Maxwell's Equations when $\rho = 0$ + $\vec{J} = 0$ (ie. either in free space or in linear dielectric media w/no free charges or currents).
- We found that we would have simple travelling wave solutions and that we could use Boundary Conditions to find Reflection & Transmission coefficients at the boundaries.

But, what happens if the material is conducting?

- The dominant physics is no longer, "$\vec{E}$ acting on bound charges" → that does still happen.
- But, more importantly $\vec{E}$ interacts with free charges!

A complete, general understanding of this is quite complicated, so we will make a few simplifying assumptions. We will still obtain some very cool results including:

- Metals are shiny (R is generally large)
- Metals are opaque
- Metals have a small skin depth into which $\vec{E}$ fields penetrate
- EM Waves travel slower in metals than we might guess
- $\vec{B}$ & $\vec{E}$ are not in phase in metals (which is the origin of radiation pressure)
- $\vec{B}$ dominates inside metals (most energy is in $\vec{B}$)

This all will come from Maxwell's Eqns + B.Cs

Statically, we know that $\vec{E}=0$ and $\rho_f=0$ inside of metals. But what about if the electric field oscillates? Can you build up charges locally?

Suppose you do: Imagine dumping $\rho_0 \neq 0$ deep inside the metal at $t=0$, what happens? We know it will tend to repel itself and head to the edges. How long will that take?

→ We can get a rough estimate from charge conservation and Ohm's Law.

$$\nabla\cdot\vec{J} = -\partial\rho/\partial t \qquad \vec{J} = \sigma\vec{E}$$

conductivity, not surface charge.

$$\nabla\cdot(\sigma\vec{E}) = -\partial\rho/\partial t \underset{\text{homogeneous conductor}}{\Rightarrow} \sigma(\nabla\cdot\vec{E}) = -\partial\rho/\partial t$$

$$\sigma\rho/\epsilon_0 = -\partial\rho/\partial t$$

We are left with $\partial\rho/\partial t = -\frac{\sigma}{\epsilon_0}\rho$.

which has the solution: $\rho(t) = \rho_0 e^{-\sigma/\epsilon_0 t}$

So it dissipates exponentially fast!

It's "gone" in a time $\simeq t_{ohmic} = \epsilon_0/\sigma = \frac{8.85\cdot10^{-12}}{10^8} \simeq 10^{-19}\,s$

So for any $t \gg t_{ohmic}$, we will have $\boxed{\rho_0 = 0 \text{ in conductors}}$

this jives with our intuition from Phy 481, where $\rho=0$ everywhere in metals!

If you take into account the fact that bound charges can polarize, just replace $\nabla \cdot \vec{E} = \rho/\epsilon_0$ with $\nabla \cdot \vec{D} = \rho_f$.

For linear media, $\nabla \cdot \vec{D} = \nabla \cdot (\epsilon \vec{E}) = \rho_f$.

- The free charges are the only ones that can run away, so this gives a slightly longer time $= \epsilon/\sigma$.
- this makes sense because the polarizing atoms in the lattice reduce the internal E fields, which are responsible for the dissipation of charge to the surface

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there's another time scale to consider with metals that's "hidden" in Ohm's Law, $\vec{J} = \sigma \vec{E}$.

Ohm's Law arises from impurities and collisions, which have their own time scales,

$$\tau_{\text{collision}} \simeq \frac{\text{interaction distance}}{\text{typical } e^- \text{ speed}} \cong 10^{-14}\,\text{s} \quad \text{for most metals.}$$

So, if we consider high frequency $\vec{E}$ fields then life gets a bit complicated.

But if $f = \frac{1}{T} \ll \begin{cases} 1/\tau_{\text{ohmic}} \sim 10^{19}\,\text{Hz} \\ \& \ 1/\tau_{\text{collision}} \sim 10^{14}\,\text{Hz} \end{cases}$ then we're ok!

a) $f \ll \frac{1}{\tau_{\text{ohmic}}}$ let's us use $\rho_f = 0$ because there's plenty of time for charge to dissipate.

b) $f \ll \frac{1}{\tau_{\text{collision}}}$ lets us use Ohm's Law, $\vec{J} = \sigma \vec{E}$ because there's plenty of time for collisions to occur giving rise to this model.

For $f \lesssim 10^{14}$ Hz, we should be good to go. This includes radio, TV, microwaves, IR, and almost visible. (But, e.g. X-rays & Gamma Rays really require new analysis)

Maxwell's Equations for our analysis,

$$\nabla \cdot \vec{E} \simeq 0 \qquad (\rho \simeq 0)$$

$$\left.\begin{array}{l} \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} = -\dfrac{d\vec{B}}{dt} \end{array}\right\} \text{always!}$$

$$\nabla \times \vec{H} = \vec{J}_f + \frac{d\vec{D}}{dt} \Rightarrow \underset{\text{homogenous media}}{\text{assume linear}} \Rightarrow \nabla \times \vec{B} = \mu \vec{J}_f + \mu \varepsilon \frac{d\vec{E}}{dt}$$

This set of equations is nearly identical to our dielectric story (with $\rho_f = 0$ assumed). There is one correction, we have an extra term: $\mu \vec{J}_f = \mu \sigma \vec{E}$ (Ohm's)

→ We cannot set $\vec{J}_f = 0$ here!!

- Metals conduct, there are currents and this makes all the difference.
- For good conductors (big σ), this new term will dominate the physics.

So our final Maxwell equation is,

$$\nabla \times \vec{B} = \underbrace{\mu \sigma \vec{E}}_{} + \mu \varepsilon \, d\vec{E}/dt$$

↳ this is what is new

(not appearing in empty space or dielectric media with $\rho_f = 0$)

Let's setup a situation like we have in the past where we send EM Waves into a metal from another linear homogeneous material.

→ from this new term Ohmic currents will appear in the metal, which dissipate the energy of the wave.

→ Free travelling waves should not be our solution because the energy will dissipated (thermal energy increase of metal)

→ $\vec{E}$ will dissipate within some distance (skin depth effect)

<u>Lets see how</u> by <u>starting</u> <u>with</u> <u>Maxwell's</u> <u>Eqn's</u>.

$$\nabla\times(\nabla\times\vec{E}) = -\frac{\partial}{\partial t}(\nabla\times\vec{B})$$

$$\nabla(\underbrace{\nabla\cdot\vec{E}}_{0 \text{ for } t \gg t_{ohmic}}) - \nabla^2\vec{E} = -\left(\mu\sigma\frac{\partial\vec{E}}{\partial t} + \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t}\right)$$

results in the following wave equ;

$$\nabla^2\vec{E} = \underbrace{\mu\sigma\frac{\partial\vec{E}}{\partial t}} + \mu\varepsilon\,\partial^2\vec{E}/\partial t^2$$

this is new and b/c σ is large for good conductors, this will Matter!

Can we solve this equation?

→ We can always try something so let's choose our travelling wave solution and see what happens.

$$\tilde{\vec{E}} = \tilde{\vec{E}}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

To make things a touch easier to follow, let's choose $\hat{k} = \hat{z}$

So, $\tilde{\vec{E}} = \tilde{\vec{E}}_0 e^{i(kz - \omega t)}$

$\nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$ gives us,

$-k^2 \tilde{\vec{E}} = -i\omega\mu\sigma \tilde{\vec{E}} - \omega^2 \mu\varepsilon \tilde{\vec{E}}$ (each term still contains $\tilde{\vec{E}}$)

So, we can cancel out the $\tilde{\vec{E}}$'s,

$$K^2 = \underbrace{i\omega\mu\sigma}_{\text{comes from } \vec{J}_f} + \underbrace{\omega^2\mu\varepsilon}_{\text{comes from displacement current.}}$$ [We have a complex wave number now!]

So our solution will work as long as K satisfies this

(we will rename this $\tilde{k}$ to remind us that it is complex)

this equation is actually familiar, in vacuum $\sigma = 0$, so that $K^2 = \mu_0 \varepsilon_0 \omega^2$, which told us that $V = \frac{k}{\omega} = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = c$!

Comment 1: $\nabla \cdot \vec{E} = 0$ tells us that $k(\tilde{\vec{E}}_0)_z = 0$ or more generally that $\vec{k} \cdot \tilde{\vec{E}}_0 = 0$ so that our solution is again transverse. $\vec{E}_0$ is $\perp$ to $\vec{k}$.

Comment 2: Taking the curl of Ampere-Maxwell will gives us the exact same general solution for $\vec{B}$ with the exact same $\tilde{\vec{k}}^2$.

Comment 3: $\nabla \cdot \vec{B} = 0$ tells us that $\vec{k} \cdot \tilde{\vec{B}}_0 = 0$ so that B is also transverse. $\tilde{\vec{B}}_0$ is $\perp$ to $\vec{k}$.

We need $\tilde{k}$ not $\tilde{k}^2$ in our solution $\tilde{\vec{E}} = \tilde{\vec{E}}_0 e^{i(\tilde{k}z-\omega t)}$.

Taking the square root of a complex # is slightly tricky. Think about it.

<u>Method 1</u>: write $\tilde{k}^2 = |\text{Magnitude}| e^{i\theta}$ (you can write any complex # like this) and then take $\sqrt{\ }$:

$$\tilde{k} = \sqrt{|\text{Magnitude}|}\, e^{i\theta/2}$$

<u>Method 2</u>: Write $\tilde{k}^2 = a + bi$ here $a = \mu\varepsilon\omega^2$ & $b = \mu\sigma\omega$

If $\tilde{k} = k_R + k_{Im} i$, then $\tilde{k}^2 = k_R^2 - k_{Im}^2 + 2k_R k_{Im} i$

This gives us two equations and two unknowns (k_R & k_{Im})

Griffiths just writes down the solutions,

$$k_R = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\sigma/\varepsilon\omega\right)^2}+1\right]^{1/2}$$

$$k_{im} = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\sigma/\varepsilon\omega\right)^2}-1\right]^{1/2}$$

these are in general what we will need to use, but we're going to make one ~~more~~ simplifying assumption: we have a <u>very good</u> <u>conductor</u>.

[This is ok for many metals and it simplifies our mathematics to make the conceptual ideas easier to follow, so that's our rationale]

How big is σ? We will assume $\boxed{\sigma \gg \varepsilon\omega}$

So that $\sigma\mu\omega \gg \mu\varepsilon\omega^2$ which means that

$$\tilde{k}^2 = \underbrace{i\sigma\mu\omega}_{\text{big } \sigma \text{ means this dominates}} + \underbrace{\omega^2\mu\varepsilon}_{\text{neglect}} \cong (\sigma\mu\omega)i$$

($\vec{J}_f$ dominates over the displacement current.)

We are already limiting ourselves to $\omega << \sigma/\epsilon$, even for a good conductor. So $\sigma/\epsilon \simeq 10^{19}\,Hz$

so still plenty of physics here to look at and consistent with our previous assumptions.

In this limit,

$$\tilde{k} = \sqrt{\sigma\mu\omega}\sqrt{i} = \sqrt{\sigma\mu\omega}\sqrt{e^{i\pi/2}} = \sqrt{\frac{\sigma\mu\omega}{2}}\,(1+i)$$

[Quick check: $\left(\frac{1+i}{\sqrt{2}}\right)^2 = \frac{1+2i-1}{2} = i$ ✓]

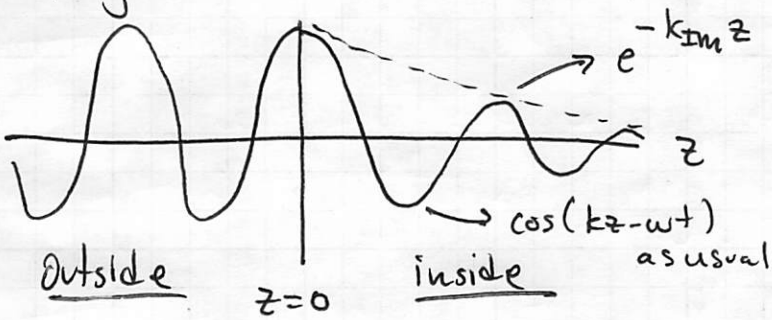
So we find that $\tilde{k} = k_R + k_{Im}\, i = \sqrt{\frac{\mu\sigma\omega}{2}} + \sqrt{\frac{\mu\sigma\omega}{2}}\, i$

And our final result for the electric field is,

$$\vec{\tilde{E}} = \vec{\tilde{E}}_0\, e^{i(\tilde{k}z-\omega t)} = \vec{\tilde{E}}_0\, e^{-k_{Im}z}\, e^{i(k_R z-\omega t)}$$

(Constant transverse vector out front) * (dying exponential) * (usual travelling wave with $v = \omega/k_R$)

The physical $\vec{E}$ is the real part of this (Reminder!!)



envelope kills off the amplitude as we enter the metal.

The wave dies out ($1/e$ factor) in a distance,

$$d \simeq \frac{1}{k_{Im}} \equiv \text{"Skin depth"}$$

In our good conductor case $d \simeq \sqrt{\frac{2}{\mu\sigma\omega}}$

It depends on frequency!

How big is that skin depth?

For normal metals with $\sigma \simeq 10^8$ and $\mu \simeq \mu_0 = 4\pi \cdot 10^{-7}$, we can find the skin depth for different frequencies,

Radio waves ($\omega \sim 10^6$) MHz gives us $d = \sqrt{\dfrac{2}{(4\pi \cdot 10^{-7})(10^8)(10^6)}}$

$d \simeq 10^{-4}\,m$ (fraction of a mm.)

Optical (visible) ($\omega \sim 10^{15}$) [which is pushing the limit of the validity of our collision time assumption]

$$d \simeq \sqrt{\frac{2}{(4\pi \cdot 10^{-7})(10^8)(10^{15})}}$$

$\simeq$ few nanometers (~ 10 atomic distances)

Conclusions: Metals have a small skin depth for any frequencies that we are likely to be interested in. So even a thin coat of metal will "act like a conductor" for EM waves. And even a micron of Al will "block" EM waves, so Aluminum foil is opaque to visible light!

Meanwhile, $k_R = \sqrt{\dfrac{\mu\sigma\omega}{2}}$ and the speed v of our travelling wave (called "v_p" for "phase velocity") we will come back to this. is ω/k_R. $v_p = \omega/k_R = \omega\sqrt{\dfrac{2}{\mu\sigma\omega}} = \sqrt{\dfrac{2\omega}{\sigma\mu}}$

Let's compare this to c: $v_p/c = \sqrt{\dfrac{2\omega}{\sigma\mu}}\sqrt{\mu_0\varepsilon_0} \simeq \sqrt{\dfrac{2\omega\varepsilon_0}{\sigma}}$

for good conductors $\sigma \gg \omega\varepsilon$ so $v_p \ll c$!

- EM waves travel slower than c, by a lot in good conductors
- Different $\omega \to$ different v_p! (Dispersion, we will return to this)

Lorentz Force

If $\vec{E} = E_0 \hat{x} e^{i(kz-\omega t)}$ $\qquad q\vec{E} = m\ddot{\vec{x}}$

Start with $z=0$, try $x = x_0 e^{i\omega t}\hat{x}$

So,

$$qE_0 = -m\omega^2 x_0 \Rightarrow x_0 = \frac{qE_0}{m\omega^2} e^{i\pi}$$

and $\vec{v} = \dot{\vec{x}} = -i\omega\vec{x} \Rightarrow v_0 = \frac{qE_0}{m\omega^2} e^{i\pi/2}$

Also, $\vec{B} = \frac{E_0}{v} e^{i\pi/4} \hat{y} e^{i\omega t}$ $\left(e^{i\pi/4} \text{ for a good conductor} \right)$

$$B_0 = \frac{E_0}{v} e^{i\pi/4}$$

$$\vec{F}_{mag} = q\vec{v}_{real} \times \vec{B}_{real}$$

$$= \frac{q^2 E_0^2}{m\omega v} \cos(kz - \omega t + \pi/2)\cos(kz-\omega t + \pi/4)\,\hat{z}$$

Note: $\langle \cos(\omega t - \pi/2)\cos(\omega t - \pi/4)\rangle$

$$= \langle \sin\omega t \tfrac{1}{\sqrt{2}}(\cos\omega t + \sin\omega t)\rangle = \frac{1}{2}\frac{1}{\sqrt{2}}$$

Had there been no $\pi/4$, we'd have $\langle \sin\omega t \cos\omega t\rangle = 0$!

What about the Magnetic field?

We claimed that taking the curl of the Maxwell-Ampere Law gives the same general solution,

$$\tilde{\vec{B}} = \tilde{\vec{B}}_0 e^{i(\tilde{k}z-\omega t)} \qquad (\text{same form w/ same } \tilde{k})$$

If we invoke Faraday's Law, we find,

$\nabla\times\vec{E} = -d\vec{B}/dt$ gives $i\tilde{\vec{k}}\times\tilde{\vec{E}}e^{i(\tilde{k}z-\omega t)} = -(i\omega)\tilde{\vec{B}}_0 e^{i(\dots)}$

So that,

$$\tilde{\vec{B}}_0 = \frac{\tilde{\vec{k}}\times\tilde{\vec{E}}_0}{\omega}$$

which is very similar to what we had before except that $\tilde{k}$ is complex now!

We are used to $B_0 = E_0/v$ as $v = \omega/k$, but now,

$$|B_0| = \frac{|E_0|}{\omega/|\tilde{k}|}$$

Using $|\tilde{k}| = \sqrt{k_R^2 + k_{Im}^2} = \sqrt{\mu\sigma\omega}$, we get $|B_0| = |E_0|\sqrt{\frac{\mu\sigma}{\omega}}$

$$= \frac{|E_0|}{c}\sqrt{\frac{\sigma}{\varepsilon_0\omega}}$$

We note that a good conductor will have

$\sigma \gg \omega\varepsilon_0$ so that for a given E_0, B_0 will be much larger than you'd find in vacuum.

So much more energy is stored in B^2, as well.

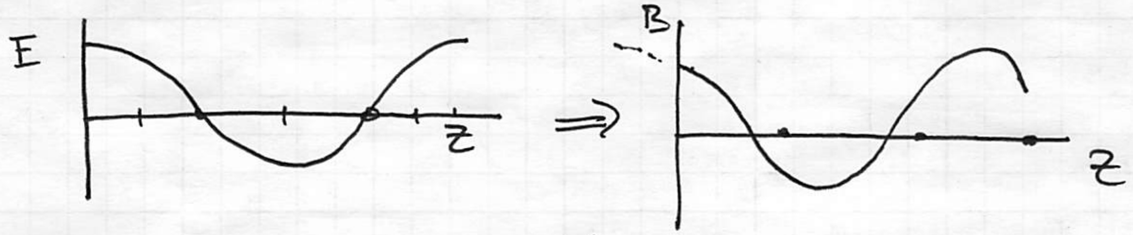
In vacuum $\vec{B}$ arises purely from J_D, but now $\vec{J}_f$ dominates the physics $J_f \gg J_D$ so the B field comes from real physical currents produced by $\vec{E}$.

Because $\tilde{\vec{B}}_0 = \dfrac{\tilde{\vec{k}} \times \tilde{\vec{E}}_0}{\omega}$ has a complex $\tilde{k}$, there in general will be a phase difference between $\vec{E}$ & $\vec{B}$ that we've not had up to now.

In the good conductor limit, $\tilde{k} = \sqrt{\mu\sigma\omega}\,\frac{1+i}{\sqrt{2}} = \sqrt{\mu\sigma\omega}\, e^{i\pi/4}$

So $\vec{B}$ is shifted by $\pi/4$ (1/8 of a cycle) from $\vec{E}$.

So if $\vec{E} \propto \cos(k_R z - \omega t)$ then $\vec{B} \propto \cos(k_R z - \omega t + \pi/4)$



B lags behind E by 1/8 of a cycle.

This phase shift has an important physical consequence. $\vec{E}$ drives free electrons, which move in the $\hat{E}$ direction. As they move the experience a $q\vec{v} \times \vec{B}$ force in the $\hat{v} \times \hat{B} = \hat{k}$ direction.

If $\vec{E}$ & $\vec{B}$ were perfectly in phase, $\vec{v}$ & $\vec{B}$ would be $90°$ out of phase (By Newton's 2nd,

$q\vec{E} = m\dot{\vec{v}}$ and one derivative brings down a i from $e^{i\omega t}$, which is a $90°$ phase)

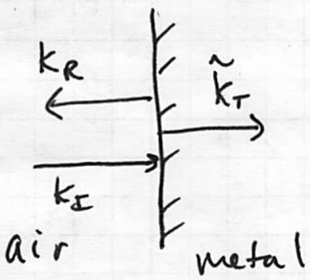
Thus, you wouldn't get no time average Lorentz force.

But with the lag you get a nonzero $\langle F_{Lorentz} \rangle$

This is the physical origin of radiation pressure.

Reflection off Metals (Normal Incidence)

→ We will focus on normal incidence for the sake of simplicity and brevity.



Boundary Conditions:

$$\begin{cases} \varepsilon_1 E_1^\perp - \varepsilon_2 E_2^\perp = \sigma_f \\ B_1^\perp - B_2^\perp = 0 \\ E_1^\parallel = E_2^\parallel \\ B_1^\parallel/\mu_1 = B_2^\parallel/\mu_2 \end{cases}$$

fortunately these are irrelevant for normal incidence, σ_f won't matter

← If no surface currents, see below

Our B.C.s here look very much the same for normal incidence as what we got for the dielectric. So the result will carry over,

$$\tilde{E}_{0R} = \tilde{E}_{0I}\left(\frac{n_1 - n_2}{n_1 + n_2}\right)$$ (we will need to think about how to define n_2, it will be complex)

* What about $\vec{K}_f$ on the surface? Griffiths points out that for an Ohmic metal, $\vec{J} = \sigma\vec{E}$ says to get a delta function surface current (infinite), you'd need an infinite $\vec{E}$ field there, which is unphysical. You get $\vec{J}$ arising from $\vec{E}$, but no singular surface currents at the edge.

** Earlier we defined $n \equiv \frac{ck}{\omega}$ so this suggests that $\tilde{n}_2 = c\tilde{k}_2/\omega$ will be complex.

there is a non-trivial phase relationship between $\tilde{E}_{0R}$ and $\tilde{E}_{0I}$!

In our good conductor, $\tilde{k}_2 = k_R + i k_{Im} = \sqrt{\frac{\mu\sigma\omega}{2}}(1+i)$

whereas k_1 is completely real, it's just ω/c.

In our case $k_R \gg \frac{\omega}{c} = k_1$ & $k_{Im} \gg k_1$

So, $\tilde{E}_{OR} = \tilde{E}_{OI}\left(\frac{k_1 - \tilde{k}_2}{k_1 + \tilde{k}_2}\right) = \left(\frac{(k_1 - k_R) - i k_{Im}}{(k_1 + k_R) + i k_{Im}}\right)\tilde{E}_{OI}$

K_1 is tiny compared to k_R & K_{Im} in the good conductor limit, so, $\tilde{E}_{OR} \simeq -\tilde{E}_{OI}$

such that $$R = \frac{|\tilde{E}_{OR}|^2}{|\tilde{E}_{OI}|^2} \simeq 1$$

so we get essentially perfect reflection from good conductors (Hence shiny metals!)

Because d is very small, even a very ~~tiny~~ thin coating of conductor is shiny. $\rightarrow$ this is how you make a mirror.