Now that we have developed the whole set of Equations that describe E+M,

$$\nabla \cdot \vec{E} = \rho/\epsilon_0 \qquad \nabla \cdot \vec{B} = 0$$

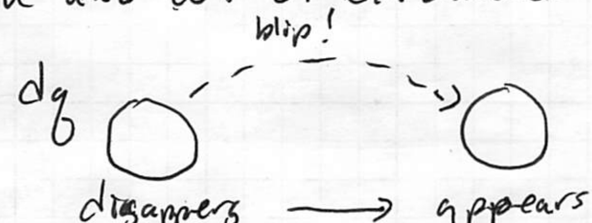
$$\nabla \times \vec{E} + \frac{d\vec{B}}{dt} = 0 \qquad \nabla \times \vec{B} - \epsilon_0 \mu_0 \frac{d\vec{E}}{dt} = \mu_0 \vec{J}$$

We will explore conservation laws in E+M.

You learned about one such conservation law → conservation of electric charge. (but there are others)

Globally: the total charge in the universe doesn't change.

It turns out this is a relatively weak statement because if it were all we know, nothing stops us from positing that charges can "blip" in and out of existence.



This would conserve charge but is not how the world works!

Locally: If a charge leaves a volume, it must flow past the boundary. (stronger statement)

We've expressed this at a point using the continuity equation $\frac{d\rho}{dt} = -\nabla \cdot \vec{J}$

increase in charge/volume = − (outflow of current density)

For a volume:

$$\underbrace{\frac{dQ}{dt} = \frac{d}{dt}\iiint_V \rho \, d\tau}_{\text{increase of charge / time}} = \underbrace{-\iiint_V \nabla\cdot\vec{J}\, d\tau = -\oiint_S \vec{J}\cdot d\vec{A}}_{-(\text{outflow of current})} = -I_{out}$$

increase of $\frac{\text{charge}}{\text{time}}$ = − (outflow of current)

So we have both global & local statements of charge conservation. Are there other local conservation laws? I think that should expect:

- Energy (we will focus on this)
- Momentum (Discuss this)
- Angular Momentum (Touch on this)

In general, "conservation of X" means that

$$\frac{dX}{dt} = -\nabla\cdot\left(\begin{array}{c}\text{volume flow of a} \\ \text{current associated w/ } X\end{array}\right)$$

Reminders about Energy

① Stored Electrical Energy $W_e = \frac{1}{2}\varepsilon_0 \iiint E^2 \, d\tau$

- work (energy) required to assemble charges to build this E field.

Electrical Energy Density $w_e = \frac{1}{2}\varepsilon_0 E^2$ (energy/volume stored in E field at a point)

② Stored Magnetic Energy $W_B = \frac{1}{2\mu_0}\iiint B^2 \, d\tau$

- work (energy) required to get currents flowing (against back EMFs) to build this B field.

Magnetic Energy Density $w_B = \frac{1}{2\mu_0} B^2$ (energy/volume stored in a B field at a point)

So the total Energy is given by,

$$U_{tot,EM} = \iiint \left(\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2\right) d\tau = \text{total stored E\&M energy in fields}$$

or

$$u_{tot,EM} = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2 = \text{stored local E\&M energy/volume}$$
$$\equiv \text{"energy density"}$$

For a statement of conservation of Energy, we are looking for a relation that looks like,

$$\frac{d}{dt}(\text{energy density}) = -(\text{outflow/vol of some energy current})$$
$$= -\nabla\cdot(\text{"energy current density"})$$

So we are going to try to figure out what this ↑ is.

- Consider some general situation with charges and currents that produce "general" $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$ fields throughout space. We will zoom in on a charge "dq" that is moving with a velocity $\vec{v}$ at a time t.
- the work done on this charge by the fields is,

$$dW_q = \vec{F}_{on\,q}\cdot d\vec{l} = \underbrace{dq\,(\vec{E}+\vec{v}\times\vec{B})}_{\text{Force on } dq}\cdot\underbrace{\vec{v}\,dt}_{d\vec{l}}$$

The magnetic field does no work so that,

$$dW_q = dq\,\vec{E}\cdot\vec{v}\,dt$$ thus the work per unit time,

$$\frac{dW_q}{dt} = dq\,\vec{E}\cdot\vec{v} = \underbrace{(\rho\, d\tau)}_{dq}\left(\vec{E}\cdot\frac{\vec{J}}{\rho}\right) \qquad \rho\vec{v}=\vec{J}$$

* Assume that we are Lorentz averaging here.

Thus, the energy per unit time is given by,

$$\frac{dW_q}{dt} = \vec{E}\cdot\vec{J}\, d\tau$$

For many charges we can express a global form,

Globally: $$\frac{dW_q}{dt} = \iiint_V \underbrace{(\vec{E}\cdot\vec{J})}_{\substack{\text{this is the E\&M power density} \\ (\text{Joules}/\text{sec}\cdot m^3)}} d\tau$$

We can also make a local statement that uses u, the energy density,

Locally: $\frac{du}{dt} = \vec{E}\cdot\vec{J}$ E&M work done on charged particles per unit volume.

- This is not expressed in the way we intended (yet) so let's explore a few vector manipulations to see if we can get it there.
- We will make use of Maxwell's Equations to reexpress the statement above in terms of $\vec{E}$ + $\vec{B}$ fields.

Start with $\nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{d\vec{E}}{dt}$

$$\vec{J} = \frac{1}{\mu_0}\left(\nabla\times\vec{B} - \mu_0\varepsilon_0\frac{d\vec{E}}{dt}\right)$$

so, $$\vec{E}\cdot\vec{J} = \frac{\vec{E}\cdot(\nabla\times\vec{B})}{\mu_0} - \varepsilon_0\vec{E}\cdot\frac{d\vec{E}}{dt}$$ (call this eqn'(a))

Here's an (inobvious) step,

$$\nabla\cdot(\vec{E}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{E}) - \vec{E}\cdot(\nabla\times\vec{B})$$ (math from product rule #6 in Griffiths)

the first term in (a) is,

$$\vec{E}\cdot(\nabla\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{E}) - \nabla\cdot(\vec{E}\times\vec{B})$$

From Faraday's Law, we know $\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$,
so we have,

$$\vec{E}\cdot\vec{J} = \frac{\vec{E}\cdot(\nabla\times\vec{B})}{\mu_0} - \varepsilon_0 \vec{E}\cdot\frac{d\vec{E}}{dt}$$

$$= \frac{\vec{B}\cdot(\nabla\times\vec{E}) - \nabla\cdot(\vec{E}\times\vec{B})}{\mu_0} - \varepsilon_0 \vec{E}\cdot\frac{d\vec{E}}{dt}$$

$$\vec{E}\cdot\vec{J} = \frac{-\vec{B}\cdot\frac{d\vec{B}}{dt}}{\mu_0} - \varepsilon_0 \vec{E}\cdot\frac{d\vec{E}}{dt} + \frac{\nabla\cdot(\vec{E}\times\vec{B})}{\mu_0}$$

Now, here's a second (inobvious) step,

$$\frac{d}{dt}\vec{A}^2 = 2\vec{A}\cdot\frac{d\vec{A}}{dt}$$ → comes from the chain rule (you can prove this!)

so, $\vec{E}\cdot\frac{d\vec{E}}{dt} = \frac{1}{2}\frac{d}{dt}(E^2)$ and $\vec{B}\cdot\frac{d\vec{B}}{dt} = \frac{1}{2}\frac{d}{dt}(B^2)$

thus,

$$\vec{E}\cdot\vec{J} = -\frac{1}{2\mu_0}\frac{d}{dt}B^2 - \frac{1}{2}\varepsilon_0\frac{d}{dt}E^2 - \frac{1}{\mu_0}\nabla\cdot(\vec{E}\times\vec{B})$$

or,

$$\vec{E}\cdot\vec{J} = -\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu_0} + \frac{\varepsilon_0}{2}E^2\right) - \nabla\cdot\underbrace{\left(\frac{\vec{E}\times\vec{B}}{\mu_0}\right)}$$

this term has a name $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$ (poynting vector)

Let's put this all together,

$$\frac{dW}{dt} \equiv \iiint \vec{E}\cdot\vec{J}\,d\tau = -\frac{d}{dt}\iiint\left(\frac{\varepsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2\right)d\tau - \iiint(\nabla\cdot\vec{S})\,d\tau$$

- the first term is U_{em}; the second can make use of the divergence theorem.

$$\frac{dW}{dt} = -\frac{d}{dt}U_{em} - \oiint \vec{S}\cdot d\vec{A}$$

So our statement of energy conservation globally is,

$$\frac{dW}{dt} = -\frac{d}{dt}U_{em} - \oiint \vec{S}\cdot d\vec{A}$$

① ② ③

In words,

① = work done on charges by EM fields

= ② decrease in energy stored in the fields minus

③ whatever energy flowed across the boundary

Does this make sense?

If no energy flows across the boundary (if ③ = 0),

$$\frac{dW}{dt} = -\frac{dU_{em}}{dt}$$ increase in particle energy = loss of stored field energy

seems ok. just energy conservation.

If ③ ≠ 0, there's another mechanism to feed energy to the particles, through $\vec{S}$.

$\vec{S}$ is the outflow of energy so negative outflow (inflow) yields positive work on charges.

$\vec{S} = \dfrac{\text{energy flow}}{\text{per unit time } (\text{+ area})}$ transported by $\vec{E}$ & $\vec{B}$ $= \dfrac{\vec{E}\times\vec{B}}{\mu_0}$

A local statement of energy conservation looks at densities,

Locally: $\dfrac{du_q}{dt} = \vec{E}\cdot\vec{J} = -\dfrac{\partial}{\partial t}u_{em} - \nabla\cdot\vec{S}$

this is poynting's theorem (derived in 1884)

We can reorganize this statement,

$$\frac{\partial}{\partial t}\left(u_q + u_{em}\right) = -\nabla\cdot\vec{S}$$

this is Griffith's u_{mech}, particle's energy density (could be complicated KE obviously & thermal and other forms of PE)

this is the energy density of the $\vec{E}$ & $\vec{B}$ fields

this is the $\dfrac{\text{out flow}}{\text{volume}}$ of energy current. $\vec{S} = \vec{E}\times\vec{B}/\mu_0$

the statement,

$\dfrac{\partial}{\partial t}\left(u_q + u_{em}\right) = -\nabla\cdot\vec{S}$ is our classic conservation law structure

$$\frac{\partial}{\partial t}(\text{something}) = -\nabla\cdot\left(\text{that something's associated current density}\right)$$

$\vec{S}$ energy current ~~den~~ density $= \dfrac{\text{flow of energy}}{\text{sec}\cdot\text{m}^2}$

Compare this to,

$$\frac{\partial}{\partial t}(\rho) = -\nabla\cdot\vec{J} \qquad \vec{J} = \frac{\text{flow of charge}}{\text{sec}\cdot\text{m}^2}$$

<u>Globally</u>: (integrating over a volume) we get back to

$$\frac{d}{dt}\iiint (u_q + u_{em})\,d\tau = -\iiint \nabla\cdot\vec{S}\,d\tau = -\oiint \vec{S}\cdot d\vec{A}$$

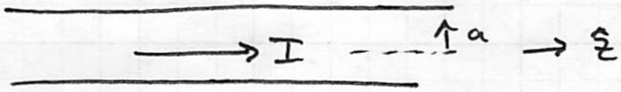
rate of increase of <u>all</u> energy = − (outflow of energy / second)

<u>Side note:</u>

In materials $\vec{S} = \vec{E}\times\vec{H}$ and

$$u_{em} = \frac{1}{2}\vec{E}\cdot\vec{D} + \frac{1}{2}\vec{H}\cdot\vec{B}$$

Example: Steady current in a wire



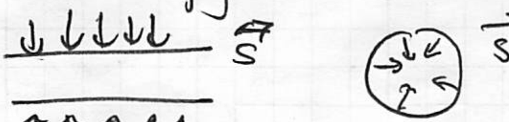
Consider a long wire with a steady current.

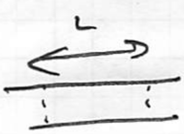
We know that $\vec{E} = E_0 \hat{z}$ and $\vec{J} = \sigma \vec{E} = \sigma E_0 \hat{z}$

As we have done in the past $\vec{B}_{inside} = \frac{\mu_0 J \pi r^2}{2\pi r}\hat{\varphi} = \frac{\mu_0 \sigma E_0}{2} r \hat{\varphi}$

At the edge $\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0} = \frac{\sigma E_0^2}{2} a \underbrace{(\hat{z}\times\hat{\varphi})}_{-\hat{s}}$

So the energy flows inwards!





Consider some length of wire, L

Across this length, $\Delta V = E_0 L$ and $I = J\pi a^2 = \sigma E_0 \pi a^2$

So, with $\frac{d}{dt}(W + U_{em}) = -\oiint \vec{S}\cdot d\vec{A}$

U_{em} is steady b/c neither $\vec{E}$ nor $\vec{B}$ change with time,

$\frac{dU_{em}}{dt} = 0$ so,

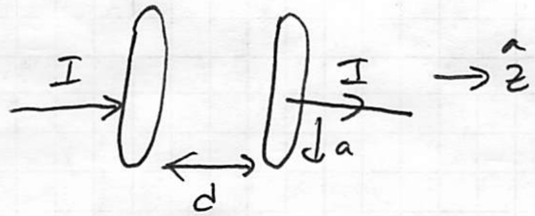
$$\frac{dW}{dt} = -\oiint \vec{S}\cdot d\vec{A} = -\frac{\sigma E_0^2}{2} a(\hat{s})\cdot(\underbrace{2\pi a L}_{\text{outer area}}\hat{s})$$

(note: end caps don't contribute)

$$= +\underbrace{(\sigma E_0 \pi a^2)}_{\text{current, } I}\underbrace{(E_0 L)}_{\text{potential diff, } \Delta V}$$

The total power entering the wire is $P = I\Delta V$!
as we've always said. It enters via the fields!
It's converted to W (U_{mech}) $\rightarrow$ thermal energy.

Example: A slowly (quasi-static) charging capacitor



we are going to investigate the energy as the capacitor charges up.

with $d \ll a$,
By Gauss' Law $\vec{E} = \frac{Q}{A\varepsilon_0}\hat{z}$ (and zero outside, right?)

By the Maxwell-Ampere Law, the magnetic field due to the wire is,

$$\oint \vec{B}(s)\cdot d\vec{l} = \mu_0 I \Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\varphi}$$

At the edge of the capacitor we showed that

$$\oint \vec{B}(s)\cdot d\vec{l} = \mu_0 \varepsilon_0 \iint \vec{J}_D \cdot d\vec{A} \quad \text{gave us}$$

$$\vec{B}(a) = \frac{\mu_0 I}{2\pi a}\hat{\varphi}$$

so the fields match there! remember?

At the edge of the capacitor ($s=a$),

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}\frac{Q}{A\varepsilon_0}\frac{\mu_0 I}{2\pi a}\underbrace{[\hat{z}\times\hat{\varphi}]}_{-\hat{s}}$$

so,

$$\vec{S} = \frac{-Q}{A\varepsilon_0}\frac{I}{2\pi a}\hat{s}$$

energy flows in as we charge!

The total energy out / time is,

$$\oiint \vec{S}\cdot d\vec{A}$$

this integral is taken in cylindrical coordinates just outside the capacitor.

$d\vec{A} = s\,d\phi\,dz\,\hat{s}$ (area points outward)
at surface of capacitor edge $s=a$

$$\oiint \vec{S} \cdot d\vec{A} = \frac{QI}{2\pi\varepsilon_0 aA} (-\hat{s}) \cdot \underbrace{(2\pi a d)}_{\text{just the outer area}} \hat{s}$$

$$= -\frac{QI}{\varepsilon_0} \frac{d}{A}$$

So the energy flows <u>into</u> the capacitor from external fields.

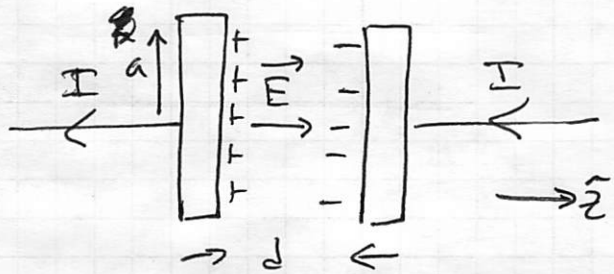
The stored energy between the plates is

$$U_{em} = \Big(\frac{1}{2}\varepsilon_0 \underset{\substack{\uparrow \\ \text{constant} \\ \text{field.}}}{E^2}\Big) \text{Volume} = \frac{1}{2}\varepsilon_0 \left(\frac{Q}{A\varepsilon_0}\right)^2 (Ad)$$

So $\dfrac{dU_{em}}{dt} = \dfrac{2Q}{2\varepsilon_0} \dfrac{dQ}{dt} \dfrac{d}{A} = \dfrac{QI}{\varepsilon_0} d/A$ which is $\oiint \vec{S} \cdot d\vec{A}$!

increase of stored energy / sec $= \dfrac{\text{flow of energy in}}{\text{sec.}}$

Example! A discharging Capacitor



A capacitor is connected to very long leads. It has a circular cross section, radius, a, and a separation, d. with $d \ll a$.

We intend to find $\vec{S}$ to see how the energy is transported.

Between the plates $\vec{E} = \dfrac{Q}{\pi a^2 \varepsilon_0}\,\hat{z}$ like usual for a capacitor.

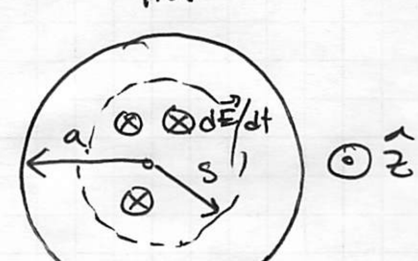
But now, $\dfrac{d\vec{E}}{dt}$ points in $-\hat{z}$! see why?

this also makes sense from a conservation of charge situation, $\dfrac{dQ}{dt} = -I$

Ok so we can compute $\vec{J}_D$,

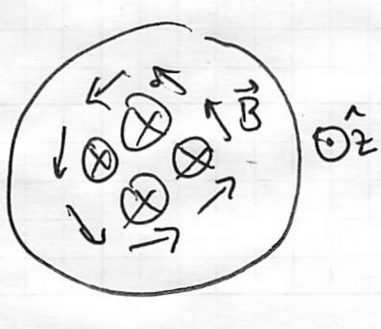
$$\vec{J}_D = \varepsilon_0 \frac{d\vec{E}}{dt} = \frac{dQ/dt}{\pi a^2}\,\hat{z} = -\frac{I}{\pi a^2}\,\hat{z}$$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \iint \vec{J}_D \cdot d\vec{A}$$

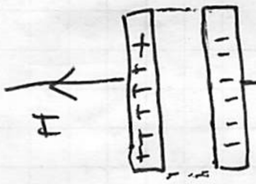


$$B 2\pi s = -\frac{\mu_0 I}{\pi a^2}\pi s^2$$

so, $\vec{B} = -\dfrac{\mu_0 I s}{2\pi a^2}\,\hat{\phi}$ circulates opposite our other example.



Makes sense b/c $d\vec{E}/dt$ points the other way.



$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

We will evaluate this at the surface of the dashed cylinder to see what the energy density current is doing.

$$\vec{S} = \frac{1}{\mu_0}\left(\frac{Q}{\pi a^2 \varepsilon_0} \hat{z} \times - \frac{\mu_0 I s}{2\pi a^2} \hat{\phi} \right)\Bigg|_{s=a}$$

$$= \frac{Q I a}{2(\pi a^2)^2 \varepsilon_0} \Big(\underbrace{\hat{z} \times - \hat{\phi}}_{+\hat{s}} \Big) = \frac{Q I a}{2(\pi a^2)^2 \varepsilon_0} \hat{s}$$

energy flows out of the region!

this is not Quasistatic! If RC circuit,

then $I(t) = \frac{V_0}{R} e^{-t/RC}$ and

$$Q(t) = C V_0 e^{-t/RC} \qquad \text{note: } C = \frac{A\varepsilon_0}{d}$$

$$\vec{S} = \frac{\left(\frac{V_0}{R}\right)\left(e^{-t/RC}\right)\left(\frac{A\varepsilon_0}{d} V_0\right)\left(e^{-t/RC}\right) a}{2A^2\varepsilon_0} \hat{s}$$

$$\vec{S} = \frac{V_0^2}{R} \frac{a}{2Ad} e^{-2t/RC} \hat{s} \qquad \tau = \frac{RC}{2}$$

energy dissapation has time constant that is 1/2 that of I or Q.

B/c not Quasistatic

$$U_{em}(t) = \iiint \frac{\varepsilon_0}{2} E^2 d\tau + \iiint \frac{1}{2\mu_0} B^2 d\tau$$

is needed to find dU_{em}/dt!

Momentum Conservation

Consider a small charge that is in a E&M field $\vec{E}$ & $\vec{B}$. This charge experiences a force,

$$\vec{F}_{on\,q} = q(\vec{E}+\vec{v}\times\vec{B})$$

we are interested in the momentum so we look at Newton's 2nd Law,

$$\vec{F} = \frac{d\vec{p}}{dt}$$

$$\frac{d\vec{p}}{dt} = q(\vec{E}+\vec{v}\times\vec{B})$$

if we think of the charge as resulting from some charge density ρ then the force on this is related to the momentum density, $\vec{\mathbb{P}}$

$$\vec{F}_{on\,q} = \iiint \rho\,(\vec{E}+\vec{v}\times\vec{B})\,d\tau = \frac{d}{dt}\iiint \vec{\mathbb{P}}\,d\tau$$

where $\vec{p} = \iiint \vec{\mathbb{P}}\,d\tau$ and thus the time rate of change of the momentum density is,

$$\frac{d}{dt}\vec{\mathbb{P}} = \rho(\vec{E}+\vec{v}\times\vec{B}) = \rho\vec{E}+\rho\vec{v}\times\vec{B} \text{ or,}$$

$$\frac{d\vec{\mathbb{P}}}{dt} = \rho\vec{E}+\vec{J}\times\vec{B}$$

in this case, the magnetic force doesn't vanish

We could go through a ton of math (in Griffiths) to eliminate ρ and $\vec{J}$ to find,

$$\frac{d}{dt}\left(\begin{matrix}\text{momentum}\\ \text{thing}\end{matrix}\right) = -\nabla\cdot\left(\begin{matrix}\text{momentum}\\ \text{current thing}\end{matrix}\right)$$

So what we will end up with is,

$$\frac{d}{dt}\left(\vec{\mathbb{P}} + \vec{\mathbb{P}}_{fields}\right) = -\nabla\cdot(\text{something})$$

this is the mechanical momentum density $\vec{\mathbb{P}}_{mech} = \dfrac{\text{momentum}}{\text{volume}}$

this is a vector involving $\vec{E}$ & $\vec{B}$ turns out to be rather simple $\vec{\mathbb{P}}_{fields} = \varepsilon_0\mu_0\vec{S}$

this requires a fair amount of math, but it must be the moment current density

the "something" is interesting b/c its divergence gives you a vector. A matrix or "2nd rank tensor" can do that. "Something" is the Stress-Energy tensor, $\overleftrightarrow{T}$.

A few mathematical operations with tensors

$$\vec{A}\cdot\overleftrightarrow{T} \equiv \langle A_x, A_y, A_z\rangle \cdot \begin{vmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{vmatrix}$$

= another Vector

for example, $(\vec{A}\cdot\overleftrightarrow{T})_x = T_{xx}A_x + T_{xy}A_y + T_{xz}A_z$

so that

$$\nabla\cdot\overleftrightarrow{T} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \cdot \begin{vmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{yz} & T_{zy} & T_{zz} \end{vmatrix}$$

$$(\nabla\cdot\overleftrightarrow{T})_x = \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z}$$

The Maxwell Stress Energy Tensor, $\overleftrightarrow{T}$, tells you the $\underline{\dfrac{\text{momentum flow}}{(\text{unit time})(\text{unit area})_\perp}}$ that is transported by $\vec{E}$ & $\vec{B}$.

$$\frac{d}{dt}\left(\vec{\mathbb{P}}_{mech} + \vec{\mathbb{P}}_{EM}\right) = -\nabla\cdot\overleftrightarrow{T}$$

is the local statement of momentum conservation

We can construct the global form by integrating over a volume,

$$\underbrace{\frac{d\vec{p}}{dt}}_{\substack{\text{just the}\\ \text{mechanical}\\ \text{force, } \vec{F}_{mech}\\ \text{(a vector)}}} + \underbrace{\frac{d}{dt}\iiint \varepsilon_0\mu_0\vec{S}\,d\tau}_{\substack{\text{rate of change}\\ \text{of the stored}\\ \text{Momentum in}\\ \vec{E}\,\&\,\vec{B}\text{ (a vector)}}} = -\iiint(\nabla\cdot\overleftrightarrow{T})\,d\tau$$

$$= \underbrace{-\oint\!\!\oint \overleftrightarrow{T}\cdot d\vec{A}}_{\substack{\text{The outflow of momentum}\\ \text{across the boundary}\\ (3\times3\text{ tensor}\circ\text{vector}=\text{vector!})}}$$

Summary: E&M fields <u>store momentum</u>, $\vec{P}_{em} = \iiint \varepsilon_0\mu_0\vec{S}\,d\tau$

- the momentum density is $\vec{\mathbb{P}}_{em} = \varepsilon_0\vec{E}\times\vec{B} = \varepsilon_0\mu_0\vec{S}$
- the energy-stress tensor $\overleftrightarrow{T}$ is the outflow of momentum per unit time per unit $\perp$ area.
- In a steady state situation, $\dfrac{d\vec{P}_{em}}{dt} = 0$

  so that $\dfrac{d}{dt}\vec{P}_{mech} = -\oint\!\!\oint \overleftrightarrow{T}\cdot d\vec{A}$

  the force on some volume of charges = is given by this tensor at the boundary

What is this tensor $\overleftrightarrow{T}$?

$$T_{ij} \equiv \varepsilon_0\left(E_iE_j - \tfrac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \tfrac{1}{2}\delta_{ij}B^2\right)$$

$$\delta_{ij} = \text{Kronecker delta} = \begin{cases} 1 & i=j \\ 0 & \text{otherwise} \end{cases}$$

So for example,

$$T_{xx} = \varepsilon_0\left(E_xE_x - \tfrac{1}{2}(1)E^2\right) + \frac{1}{\mu_0}\left(B_x \cancel{B_y}B_x - \tfrac{1}{2}(1)B^2\right)$$

$$T_{xx} = \frac{\varepsilon_0}{2}\left(E_x^2 - E_y^2 - E_z^2\right) + \frac{1}{2\mu_0}\left(B_x^2 - B_y^2 - B_z^2\right)$$

$$T_{xy} = \varepsilon_0\left(E_xE_y - \tfrac{1}{2}(0)E^2\right) + \frac{1}{\mu_0}\left(B_xB_y - \tfrac{1}{2}(0)B^2\right)$$

$$T_{xy} = \varepsilon_0(E_xE_y) + \frac{1}{\mu_0}(B_xB_y)$$

so you can see that,

$$T_{xz} = \varepsilon_0(E_xE_z) + \frac{1}{\mu_0}(B_xB_y)$$

$\overleftrightarrow{T}$ is fairly easy to compute, but its a touch harder to make sense of.

→ The diagonal elements act like pressures
T_{xx}, T_{yy}, T_{zz} integrated over a surface they act like forces.

→ The off-diagonal elements act like shears
For example, you can generate an x-directed force on a y-directed area element!

That's what T_{xy} does!

T_{yx} is a y-directed force on an x-directed element.

Angular Momentum in EM fields

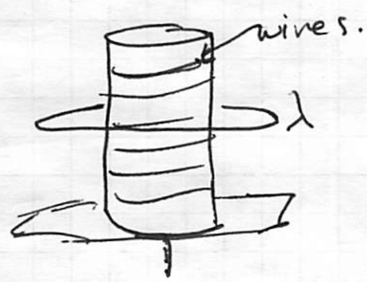
We can look at the angular momentum of charges in fields and derive an angular momentum density,

$$\vec{l}_{EM} = \vec{r} \times \vec{P}_{em} = \varepsilon_0 \vec{r} \times (\vec{E} \times \vec{B}) = \varepsilon_0 \mu_0 \vec{r} \times \vec{S}$$

EM fields (even static ones) can have linear & angular momentum! In fact, they must as they account for missing $\vec{P}$ & $\vec{L}$ in situations where $\vec{P}$ & $\vec{L}$ don't appear to be conserved, but they are!

Conceptual Example!

A solenoid rests on a platter ~~free to rotate~~ surrounded by a static ring with constant charge density λ. is free to rotate.



At $t=0$ there's no current, thus, no $\vec{B}$, everything is at rest.

We turn things on and the solenoid develops a $\vec{B}$-field. $\uparrow\uparrow\uparrow$

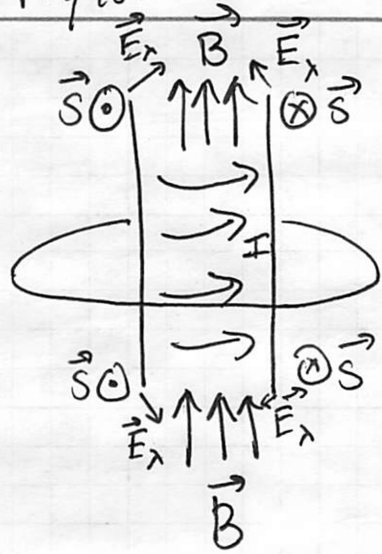
While it does so this generates a curly $\vec{E}$ field $\circlearrowleft$ so the charges in the ring feel a force and rotate!

Initially $\vec{L}_{mech} = 0$, no $\vec{B} \Rightarrow \vec{l}_{EM} = 0$ no angular momentum at start.

But after, $\vec{L}_{mech}$ is $\circlearrowleft$ not zero!

$\downarrow \vec{L}_{mech}$

In the end $\vec{B} \neq 0$, $\vec{E} \neq 0$ so there's a nonzero $\vec{l}_{em}$ up!



$\vec{r} \times \vec{S}$ points upward in every location (at least has a non-cancelling component)

$\vec{l}_{em}$ points up everywhere