

Homework 13 (Due. Apr 20)

1. Magnetic field of a wire

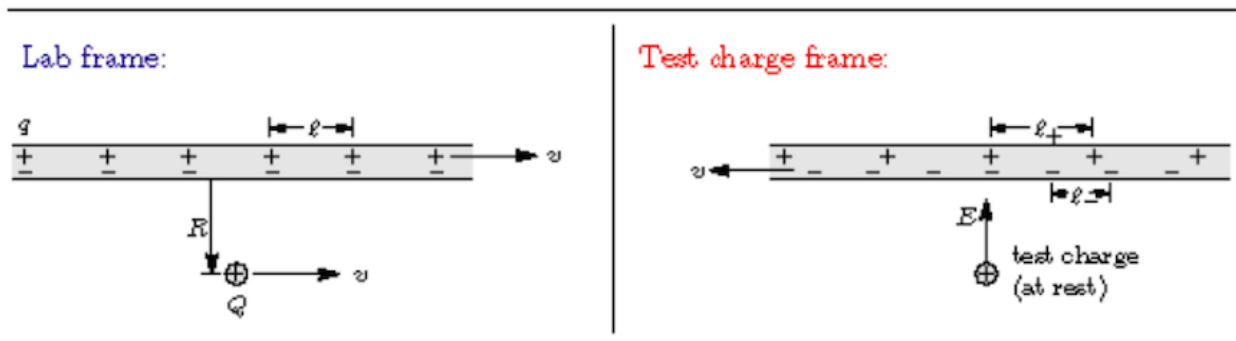


Figure 1: compare

Shown above is a model of a wire with a current flowing to the right. To avoid minus signs we take the current to consist of a flow of positive charge carriers, each with charge $+q$, separated by an average distance of l . The wire is electrically neutral in the lab frame, so there must also be a bunch of negative charges, at rest, separated by the same average distance l in this frame. Be aware that charge is Lorentz invariant: a charge Q has the same value in every inertial frame.

1. Using Gauss' law, what is the electric field outside this wire in the lab frame? Suppose there is a test charge $+Q$ outside the wire, a distance R from the center of the wire, moving to the right (For simplicity, let's say the velocity is the same as that of the moving charges in the wire, i.e. v , as shown in the figure.)
2. Given your answer for the E-field, what is the electrostatic force on this charge, in this frame?
3. Using Ampere's (and the Lorentz force) law - what is the magnetic force on the moving test charge Q ?
4. Put it together, what is the magnitude and direction of the net force on the test charge, and what "causes" it?
5. Now consider how all this looks in the reference frame of the test charge, where it's at rest. In THIS frame, what is the magnetic force on the test charge Q ? In this frame, it's the negative charges in the wire that are moving to the left. Because they're moving, the average distance between them is length-contracted. Meanwhile the positive charges are now at rest, so the average distance between them is now longer than l .
6. What is the average distance (l_+) between the positive charge carriers in this frame? Both of these effects give the wire a non-zero charge density.
7. Compute the charge density (charge per length) in this frame, with the correct overall sign.
8. Use Gauss's Law to compute the electrostatic force on the test charge.
9. In THIS frame, what is the magnitude and direction of the force on the test charge, and what "causes" it?
10. For normal currents, $\beta = v/c$ is about 10^{-13} . (Drift velocities are small!) Given this, show that the forces you computed in parts 4 and 9 are the same size. *Hint: expand in a Taylor series.*

2. E&B transformations

$$\bar{E}_x = E_x \quad \bar{E}_y = \gamma(E_y - vB_z) \quad \bar{E}_z = \gamma(E_z + vB_y)$$

$$\bar{B}_x = B_x \quad \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right) \quad \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

1. Use these equations to show that both $\mathbf{E} \cdot \mathbf{B}$ and $(E^2 - c^2 B^2)$ are Lorentz invariants.
2. We found earlier that \mathbf{E} and \mathbf{B} are mutually perpendicular for traveling EM waves. Given that this is true in some frame, can there be any other reference frame in which you would find \mathbf{E} and \mathbf{B} not perpendicular for traveling EM waves?
3. Suppose $E > cB$ in some frame. Show that there is *no* possible frame in which $E = 0$.
 - If $E = 0$ in some frame, do these relations mean that E is always equal to 0 in every other inertial frame?
 - If $B = 0$ (but E is nonzero) in some frame, can you always (ever?) find another frame in which $E = 0$ (but B is nonzero)?

3. Maxwell's equations written compactly

We found that we could write the field tensor like this

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

and the dual tensor like this,

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

With the current density 4-vector written as this: $J^\mu = (c\rho, J_x, J_y, J_z)$, we claim that Maxwell's equations in vacuum are given thusly,

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu, \quad \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$$

1. Show by explicit calculation that you can recover all 4 Maxwell's equations.
2. What would have happened if $\frac{\partial G^{\mu\nu}}{\partial x^\nu} \neq 0$? Think about what happens physically?! It must be zero!