Invariants like the proper time are important and helpful in a number of situations. Let's explore some of these a bit more.

Proper Time

Consider a particle ~~on~~ in my reference frame moving with

$u = \frac{dx}{dt}$ (with respect to me)

In my frame, the particle traverses a distance dx in a time dt.

In the particle's frame (S'), it would say that $d\tau = \frac{1}{\gamma} dt$ had elapsed (the proper time) where $\gamma = \frac{1}{\sqrt{1-u^2/c^2}}$.

Imagine a light signal that travels for that amount of time in S'.

$$c\,d\tau = c\sqrt{1-u^2/c^2}\,dt = \sqrt{c^2dt^2 - u^2dt^2} = \sqrt{c^2dt^2 - \left(\frac{dx}{dt}^2\right)dt^2}$$

$$= \sqrt{c^2dt^2 - dx^2} = \sqrt{-dx_\mu dx^\mu}$$

where dx^μ is the displacement of the particle in S.

the last form is written in the "Manifestly invariant" form. you can reexpress dx_μ in any inertial frame and $\sqrt{-dx_\mu dx^\mu}$ gives the same result!

We could have guessed this as $d\tau$ and c are both invariant quantities → it makes sense that their product is also invariant.

In general: a 4 vector multiplied by an invariant is still a 4 vector. A useful result!

Consider $\eta^\mu \equiv \dfrac{dx^\mu}{d\tau}$ $\rightarrow$ the 4vector representing displacement
$\rightarrow$ an invariant, the proper time.

This is indeed a 4vector, so it transforms quite simply,

$$\bar{\eta}^\mu = \Lambda^\mu_{\ \nu}\, \eta^\nu$$

We can determine $\bar{\eta}^\mu$ in any frame in the same ways we can find $\bar{x}$ and $\bar{t}$.

the units of this quantity suggest it's a 4 velocity (m/s)

$$\eta^0 = \frac{dx^0}{d\tau} = c\frac{dt}{d\tau} = c\gamma = \frac{c}{\sqrt{1-u^2/c^2}}$$

$$\vec{\eta} = \frac{d\vec{x}}{d\tau} = \frac{d\vec{x}}{dt}\frac{dt}{d\tau} = \vec{u}\gamma = \frac{\vec{u}}{\sqrt{1-u^2/c^2}}$$

$\rightarrow$ this is just the normal velocity.

$\hookrightarrow$ This is called the proper velocity.

$\vec{\eta}$ is a "hybrid" quantity $\vec{\eta} = \dfrac{d\vec{x}}{d\tau}$ $\rightarrow$ displacement in $\underline{\underline{S}}$
$\rightarrow$ proper time, measured in particle's rest frame.

But, η^μ is well defined and quite useful; it transforms quite simply,

$$\bar{\eta}^0 = \gamma(\eta^0 - \beta\eta^1)$$
$$\bar{\eta}^1 = \gamma(\eta^1 - \beta\eta^0)$$

in the frame $\bar{S}$ where the movement is at a speed βc in the x direction relative to S.

Note: tranforming regular velocity is nastier $d\vec{x}/dt$ because ~~$\vec{x}$ and dt~~ both $d\vec{x}$ and dt transform!

For any 4-vector, its square (scalar product with itself) is invariant, for example with η,

$$\eta^2 \equiv \eta \cdot \eta = \eta_\mu \eta^\mu = -\eta^{0\,2} + \vec{\eta}^{\,2}$$

$$= \frac{-c^2}{1-u^2/c^2} + \frac{u^2}{1-u^2/c^2} = \frac{u^2-c^2}{(c^2-u^2)/c^2} = -c^2$$

$\eta^2 = -c^2$ is very obviously invariant (true for any moving frame)

Defining η^μ is the first step in constructing more 4 vectors, which we need to build up Maxwell's theory in Relativistic form.

→ You might be interested in ~~...~~ considering one more derivative to find the "4-acceleration" or the "4-force". But first, let's consider,

<u>Four Momentum</u> $\quad p^\mu \equiv m\eta^\mu = m\frac{dx^\mu}{d\tau}$ (its mass * 4 velocity)

Here $\quad m \equiv$ <u>rest</u> <u>mass</u> of the object

→ measured in its rest frame

→ manifestly invariant

so p^μ is a 4 vector, which transforms simply,

$$\bar{p}^\mu = \Lambda^\mu_{\ \nu} p^\nu$$

$$p^0 = m\eta^0 = mc\gamma = \frac{mc}{\sqrt{1-u^2/c^2}}$$

$$\vec{p} = \frac{m\vec{u}}{\sqrt{1-u^2/c^2}}$$

In the rest frame of the particle, $\vec{u}=0$ so that,

$$\vec{p}=0 \qquad P^{\mu}_{\substack{rest\\frame}} = \begin{pmatrix} mc \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$p^0 = mc$$

In my frame, S, where the particle moves with velocity $\vec{u}$, p^0 is larger. The faster it goes the larger p^0 is! It's a scalar that increases with speed $\rightarrow$ sounds a lot like an energy!

To get the units right, let's define,

$$E \equiv cp^0 = \gamma mc^2 \qquad \text{"Relativistic Energy"}$$

In the rest frame, $E = mc^2$ "rest energy"

and

$$E - E_{rest} = (\gamma - 1)mc^2$$

(kinetic energy the part due to the motion.)

Why this classical name for $E - E_{rest}$?

If $u/c \ll 1$, then $\gamma = \dfrac{1}{\sqrt{1-u^2/c^2}} = (1-u^2/c^2)^{-1/2}$

$$\simeq 1 + \tfrac{1}{2}u^2/c^2$$

thus, $E - E_{rest} = (\gamma-1)mc^2 \simeq (1+\tfrac{1}{2}u^2/c^2 - 1)mc^2$

$$\simeq \tfrac{1}{2}mu^2 !$$

Just our old kinetic energy definition!

for very relativistic particles, $u/c \simeq 1$ so γ is huge!

$KE = mc^2(\gamma - 1) \simeq \gamma mc^2 = E_{tot}$ (rest mass energy small compared to kinetic!)

$|\vec{p}| = \gamma m u \simeq \gamma m c$

So that $|\vec{p}| \simeq E/c$ for very relativistic particles. ($|\vec{p}| \gg mc$)

In this case $p \neq mv$ and $K \neq \frac{1}{2}mv^2$.

What about the invariant quantity $p_\mu p^\mu$?

$$p_\mu p^\mu = -p^{0\,2} + \vec{p}^{\,2} = \gamma^2(-m^2c^2 + m^2u^2)$$

$$= \frac{1}{1 - u^2/c^2}\, m^2(u^2 - c^2) = -m^2c^2$$

→ $p_\mu p^\mu = -m^2c^2$ gives you a quick definition of the rest mass! (easy to find it, too.)

→ Shows that the rest mass is manifestly invariant.

Using our definition that $E \equiv cp^0$,

$$-p^{0\,2} + \vec{p}^{\,2} = -m^2c^2 \Rightarrow -\frac{E^2}{c^2} + \vec{p}^{\,2} = -m^2c^2$$

such that, $\boxed{E^2 = p^2c^2 + m^2c^4}$ (relativistic version of $\frac{p^2}{2m} = E$)

Full energy, γmc^2 (not just KE as in classical form)

When we change frames, $\bar{p}^\mu = \Lambda^\mu_{\ \nu}\, p^\nu$ mixes up $\vec{p}$ & E like space & time; but, $p_\mu p^\mu$ is always invariant.

Discussion

→ $|\vec{p}|$ and E grow (indefinitely!) with speed. But it's not the old $p \propto v$ and $E \propto v^2$ anymore!

→ Experimentally, p^μ is <u>conserved</u>

⇒ This is why we chose to construct p^μ as we did → its a useful quantity.

This means that $\sum \vec{p}_{\text{all particles}}$ is conserved

$\sum E_{\text{all particles}}$ is conserved

Conserved is <u>NOT</u> Invariant!

→ Invariance is something that is the same for all observers in all inertial frames

→ Conserved means something that <u>any</u> observer says doesn't change with time.

So, for example, E is conserved, but is <u>not</u> invariant. (it's one component of a 4vector and is different in different frames)

- m is <u>invariant</u> but <u>not</u> conserved, you can have interactions where m changes (e.g. when matter and antimatter annihilate to photons, which have zero mass.)

- Charge is invariant & conserved.

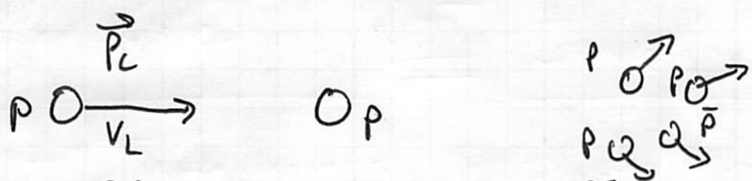
In relativistic collisions, 4 vectors can help us simplify our work → invariant quantities are very helpful!

Example: At the Bevatron, anti-protons were discovered.

$p + p \longrightarrow p + p + p + \bar{p}$ ← anti proton.

Energetic protons hit protons at rest which produce proton-antiproton pairs (and the protons you started with)

In the Lab frame:

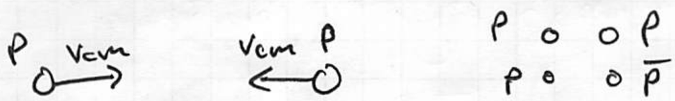


Initial

final

(What energy would be needed for the initial proton? this was part of the original design!)

In the Center of Momentum frame:



Initial

Final

(At the precise "threshold" value of $\vec{p}_L$, the particles are just created and thus at rest in the cm frame)

$p^{tot}_\mu p^\mu_{tot}$ is a frame invariant quantity (same in the lab & cm frames), but p_μ^{tot} is also a conserved quantity (same before & after the collision)

In the Lab frame:

$$P^{\mu}_{tot,init} = \underbrace{(\gamma_L mc, \vec{p}_{Lab})}_{\text{incoming proton}} + \underbrace{(mc, 0)}_{\text{target proton}}$$

In the CM frame:

$$P^{\mu}_{tot,final} = \underbrace{(4mc, 0)}$$

↳ 4 objects at rest (remember $p^0 = E/c$)

Using Invariance & Conservation together!

$$P_{\mu\,tot,init}\, P^{\mu}_{tot,init} = P_{\mu\,tot,final}\, P^{\mu}_{tot,final}$$

so that,

$$P_{lab}^2 - m^2c^2(1+\gamma_L)^2 = 0 - (4mc)^2$$

Notes: for the incoming proton $E_{lab} = \gamma_{lab} mc^2$ ← always

and $E_{Lab}^2 = p_{lab}^2 c^2 + m^2c^4$ ← always

So that

$$p_{lab}^2 = \frac{E_{lab}^2}{c^2} - m^2c^2 = \frac{(\gamma_{Lab} mc^2)^2}{c^2} - m^2c^2 = m^2c^2(\gamma_{Lab}^2 - 1)$$

With $p_{lab}^2 = m^2c^2(\gamma_{lab}^2 - 1)$,

$$m^2c^2(\gamma_L^2 - 1) - m^2c^2(1+\gamma_L)^2 = -16m^2c^2$$

or,

$$(\gamma_L^2 - 1) - (1 + 2\gamma_L + \gamma_L^2) = -16$$

$$-2 - 2\gamma_L = -16 \rightarrow 2\gamma_L = 14 \quad \gamma_L = 7$$

This tells us that we need,

$$E_{Kin,Lab} = (\gamma - 1) mc^2 \simeq 6mc^2$$

which is about 6 GeV ($m_p \simeq 938\ MeV/c^2 \sim 1\ GeV/c^2$)

The Bevatron (Billion GeV) was built specifically for this purpose → resulted in the 1959 Nobel Prize.

(Griffiths has a lot more cool examples using invariants to solve collision problems)

We've explored the concept of 4 momentum pretty fully here, so lets dig into the concept of "4-force".

Nonrelativistically, $\vec{F}_{net} = \frac{d\vec{p}}{dt}$ relates the momentum and the force.

⇒ Turns out to be relativistically correct if you use $\vec{p} = \gamma m\vec{v}$.

⇒ But because that $\frac{d}{dt}$ and not $\frac{d}{d\tau}$, this equation will not transform nicely, it's not like our other descriptions.

We will define the "Minkowski Force"

$$K^\mu = \frac{dp^\mu}{d\tau} \quad \text{which is manifestly invariant}$$

so,
$$\vec{K} = \frac{d\vec{p}}{d\tau} = \frac{d\vec{p}}{dt}\frac{dt}{d\tau} = \gamma\vec{F}$$

$$K^0 = \frac{dp^0}{d\tau} = \frac{d}{d\tau}\left(\frac{E}{c}\right) = \frac{1}{c}\left(\frac{dE}{d\tau}\right)$$ (which will be related to $\vec{F}$ & $\vec{v}$)

Note: K^μ is formally important, but I've not used it to determine motion. For that I use $\vec{F} = d\vec{p}/dt$ in a given frame w/ $\vec{p} = \gamma m \vec{v}$.

That is, $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) = d\vec{p}/dt$ is just fine with $\vec{p} = \gamma m \vec{v}$ in a given frame!

Finally, with $E^2 = p^2c^2 + m^2c^4$ we have,

$$2E\frac{dE}{dt} = 2\vec{p}\cdot\frac{d\vec{p}}{dt}c^2 + 0 = 2\vec{p}\cdot\vec{F}c^2$$

With $\vec{p} = \gamma m \vec{v}$ and $E = \gamma mc^2$, $\dfrac{\vec{p}c^2}{E} = \vec{v}$

So that,

$$\frac{dE}{dt} = \frac{\vec{p}\cdot\vec{F}c^2}{E} = \vec{F}\cdot\vec{v} = \vec{F}\cdot\frac{d\vec{x}}{dt}$$

So that $dE = \vec{F}\cdot d\vec{x}$ Work-energy thm still works with $E = \gamma mc^2$.

Also,

$$K^0 = \frac{1}{c}\left(\frac{dE}{d\tau}\right) = \frac{1}{c}(\gamma\,\vec{F}\cdot\vec{v}) = \frac{1}{c}\left(\frac{dE}{dt}\frac{dt}{d\tau}\right)$$

because