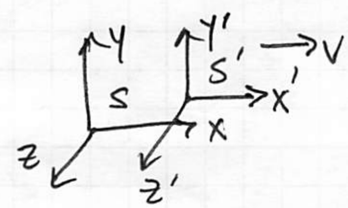
Up to now, we've used thought experiments and some "tricks" to produce the "correct" results.

It's time that we be systematic about these frame changes and understand better what we mean by the "coupling of space & time"

<u>Lorentz Transformations</u> will become our tool for understanding this a bit better. It will reproduce time dilation, length contraction, and produce a new result: velocity addition.

<u>Reminder: Galilean Transformations</u>

Consider an event that occurs at $\langle x, y, z, t \rangle$ in frame S. The same event is observed to happen at $\langle x', y', z', t' \rangle$ in frame S'. The 1D case looks like this,



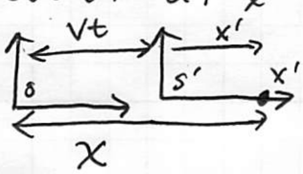
S' moves at speed v with respect to S
(or S moves at speed ~~v~~ −v with respect to S')

Let y & z match up with y' & z' for simplicity.

We set our ~~...~~ clocks to match as S' origin passes the origin of S,

$x = x' = 0$ ~~and~~ when $t = t' = 0$.

Galileo (and all my prior experiences) tell me that an event at x' in S' will be at $x' + vt$ as seen in S.



so,

$$x' = x - vt \qquad x = x' + vt$$
$$y' = y \quad \text{or} \quad y = y'$$
$$z' = z \qquad z = z'$$
$$t' = t \qquad t = t'$$

reasonable results
common sense
agree w/ ~~...~~ Newton.

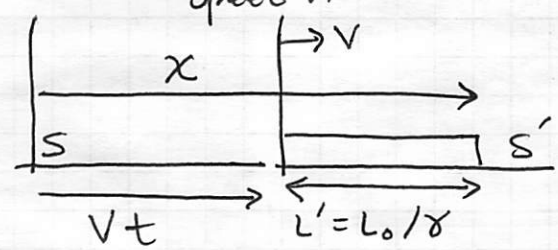
But this is wrong! Disagrees w/ time dilation, length contraction and that simultaneity is frame dependent.

Our results from length contraction & time dilation help guide us here.

Suppose that a ruler (rest length L_0) sits (at rest in S') with the left end $x'=0$ and the right end $x'=L_0$ in S'

What is observed in S?

– A length contracted ruler, $L' = \frac{L_0}{\gamma}$ moving at speed v.



An event at the right edge of the ruler is at

$$\underbrace{x}_{\text{location in } S \text{ relative to origin}} = \underbrace{vt}_{\text{movement of origin in frame } S} + \underbrace{L'}_{\text{length of stick observed in frame } S} \quad \text{in } S$$

Notice vt is still the as you would expect and all that has changed is the measure of the ruler in the frame S. it's length contracted in that frame.

In general $\quad x = vt + \frac{x'}{\gamma} \rightarrow$ in S', the stick is at rest! the right end is at x' always ind. of time!

Thus, $\quad x' = \gamma(x - vt)$

$$\text{position in moving frame} = \gamma\left(\text{position in other frame minus } v * \text{time}\right)$$

In the 1D case, $y'=y$ and $z'=z$, but what about t'?

We expect something to be different, right? (time dilation)

Consider the S' frame moving left instead,

it makes sense that $\quad x = -vt + L' = -vt + x'/\gamma$

now, just flip the sign of v. So that

$x' = \gamma(x + vt)$ again just a sign change.

But going back to the right moving S' frame, it's the S frame that is moving <u>left</u> with the same speed, v.

$\Rightarrow$ flip your perspective so that S' is the "original frame" a S is moving with $-v$. Working through this we will find,

$$x = \gamma(x' + v't)$$

Basically just
$x \leftrightarrow x'$
$t \leftrightarrow t'$
$v \leftrightarrow -v$

But γ is the same in both cases.

If (1) $x' = \gamma(x - vt)$
and (2) $x = \gamma(x' + vt)$ then plug (2) into (1)

$$x' = \gamma(\gamma(x' + vt') - vt) \text{ and solve for } t,$$

$$x' = \gamma^2 x' + \gamma^2 vt' - \gamma vt$$

$$x'(1-\gamma^2) - \gamma^2 vt' = -\gamma vt$$

or, $t = \gamma t' - \dfrac{(1-\gamma^2)}{\gamma v} x'$ which can be cleaned up,

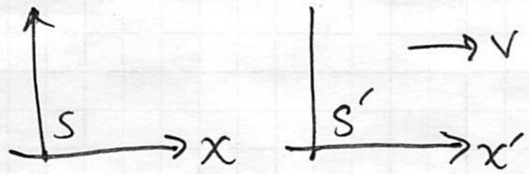
$$\frac{(1-\gamma^2)}{\gamma v} = \frac{\gamma(1-\gamma^2)}{\gamma^2 v} = \frac{\gamma}{v}\left(\frac{1}{\gamma^2} - 1\right) = \frac{\gamma}{v}\left(1 - \frac{v^2}{c^2} - 1\right)$$

$$= -\gamma v / c^2$$

so that $t = \gamma\left(t' + \frac{v}{c^2} x'\right)$

the usual trick
$x \leftrightarrow x'$
$t \leftrightarrow t'$
$v \leftrightarrow -v$ works $\Rightarrow$ $t' = \gamma\left(t - \frac{v}{c^2} x\right)$

The Complete Lorentz Transformations



$$x' = \gamma(x - vt) \qquad\qquad x = \gamma(x' + vt)$$
$$y' = y \qquad \text{or} \qquad y = y'$$
$$z' = z \qquad\qquad z = z'$$
$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \qquad\qquad t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

$$\text{with } \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Comments:

- Newton's Laws are not invariant under these transformations. We will have to fix them up.
- Maxwell's Eqn's are invariant under these transformations. No need to fix them.
- We now know how different observers measure events and we can use this see how velocity and acceleration tranform. We'll do this later.
- These rules can be summarized in matrix notation that will lead us to a powerful new notation & idea: 4vectors. We will do this soon.

⇒ We will find invariants, ones that don't depend on different frames.

Let's revist time dilation & Lorentz contraction with these new formulae:

Time Dilation: Assume one clock in S and two events happen at the location of <u>that</u> clock. Event 1 $\langle x, t_1 \rangle$
Event 2 $\langle x, t_2 \rangle$

so $\Delta t_{\text{in } S} = t_2 - t_1$

In S', $t_1' = \gamma\left(t_1 - \frac{v}{c^2}x\right)$

and $t_2' = \gamma\left(t_2 - \frac{v}{c^2}x\right)$

$$\Delta t_{\text{in } S'} = t_2' - t_1' = \gamma t_2 - \gamma t_1 = \gamma \Delta t_{\text{in } S}$$

This is time dilation, $(\Delta t)_{\text{in } S}$ is the minimum, it's the proper time.

(one clock! events happen at same x, $\Delta x = 0$)
(Note that $\Delta x' \neq 0$!)

Length Contraction: You have a stick at <u>rest</u> in S with length L_0, what does S' observe for length?

You <u>must</u> pick <u>one time</u> in t' to measure both ends (that's what is meant by observing length in S')

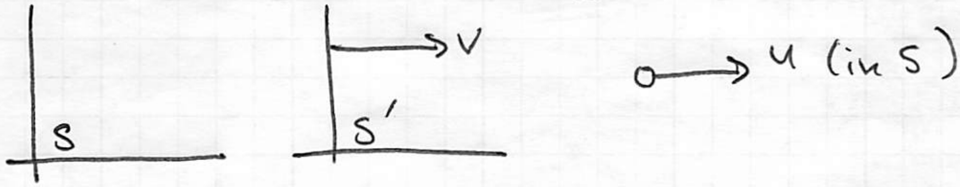
$(X_2)_{\text{in } S} = \gamma(X_2' + v t_2')$ $\quad \Delta X = X_2 - X_1$

$(X_1)_{\text{in } S} = \gamma(X_1' + v t_1')$ $\quad = \gamma(X_2' - X_1') + \gamma V \underbrace{(t_2' - t_1')}_{0}$

$\Delta X = \gamma \Delta X'$ or $\Delta X' = \frac{1}{\gamma}\Delta X$ length is shorter.

Velocity Addition

Suppose as we sit in frame S and object moves past at a speed u. What does an observer in S′ moving with speed v measure?



In S, $u = dx/dt$.

the Lorenetz Transformations give us,

$$x' = \gamma(x - vt) \Rightarrow dx' = \gamma(dx - v\,dt)$$

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \Rightarrow dt' = \gamma\left(dt - \frac{v}{c^2}dx\right)$$

where $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$

In the S′ frame the observer would observe,

$$u' = dx'/dt' = \frac{\gamma(dx - v\,dt)}{\gamma(dt - v/c^2\,dx)} = \frac{\cancel{\gamma}}{\cancel{\gamma}}\frac{dx - v\,dt}{(dt - v/c^2\,dx)}$$

$$u' = \frac{\cancel{d}t\,(dx/dt - v)}{\cancel{d}t\,\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)} = \frac{u - v}{\left(1 - \frac{v}{c^2}u\right)}$$

Einstein's Subtraction Rule

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}} \quad \text{where } u = dx/dt,\ v = \text{speed of } S' \text{ frame.}$$

If the object moved with u' in S′ and we wanted, u,

$$u_{(\text{in } S)} = \frac{u'_{\text{in } S'} + v_{(S' \text{ w.r.t. } S)}}{1 + u'v/c^2}$$

Notes: u never greater than c (we will see why)

- add velocities like this & never exceed c!
- v small? denom $\simeq 1$, $u = u' + v$
- u' or $v = c$? result is still c.