Electromagnetic Waves in Matter

Up to now, we've considered only EM waves in free space. In matter, we must start with Maxwell's equations in materials,

$$\nabla\cdot\vec{D} = \rho_f \qquad \nabla\times\vec{E} = -d\vec{B}/dt$$
$$\nabla\cdot\vec{B} = 0 \qquad \nabla\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{dt}$$

where $\vec{D} = \varepsilon_0\vec{E} + \vec{P}$ and $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$

- Lets explore what happens when $\rho_f = 0$ & $\vec{J}_f = 0$, that is when only the medium is responding to the fields and there are no free charges or currents in the region.

As before, we start with,

$$\nabla\times(\nabla\times\vec{E}) = -\frac{\partial}{\partial t}(\nabla\times\vec{B})$$
$$\parallel$$
$$\vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} = -\frac{\partial}{\partial t}(\nabla\times\vec{B})$$

Before we invoked $\nabla\cdot\vec{E} = 0$ but now we don't know if $\nabla\cdot\vec{P} = 0$ or not b/c $\nabla\cdot\vec{D} = 0$ not $\nabla\cdot\vec{E}$ now!

So we will make a few simplifying assumptions

1) Material is linearly responsive

that is, $\vec{D} = \varepsilon\vec{H}$
$\vec{H} = \vec{B}/\mu$

2) ε and μ are homogeneous constants (properties of the material)

↳ this means not a function of position!

With these two assumptions, $\nabla\cdot\vec{E} = \nabla\cdot(\vec{D}/\varepsilon) = 0$ as $\rho_f = 0$ & ε = constant!

and $\nabla\times\vec{B} = \nabla\times(\mu\vec{H}) = \mu(\nabla\times\vec{H})$

since μ is a constant

So we have,
$$0 - \nabla^2\vec{E} = -\mu\frac{d}{dt}(\underbrace{\vec{J}_f}_{0} + d\vec{D}/dt)$$
$$-\nabla^2\vec{E} = -\mu\frac{d^2\vec{D}}{dt^2}$$

Or more simply,
$$\nabla^2\vec{E} = \mu\varepsilon\frac{d^2\vec{E}}{dt^2}$$

this is just a "simple" wave equation much like our vacuum wave eqn. ($\nabla^2\vec{E} = \mu_0\varepsilon_0\frac{d^2\vec{E}}{dt^2}$), but the speed of the wave is now changed,

Before $c \equiv \frac{1}{\sqrt{\mu_0\varepsilon_0}}$ now $v = \frac{1}{\sqrt{\mu\varepsilon}}$

We'd get the same kind of equation for $\vec{B}$, but again with $v = \frac{1}{\sqrt{\mu\varepsilon}}$. But all our old results apply!

$\vec{B}\perp\vec{E}$ and both are $\perp$ to $\vec{k}$ and finally $\vec{B} = \frac{1}{v}\vec{E}$

A couple of footnotes,

1) $\varepsilon > \varepsilon_0$ always! So $\vec{E}$ fields polarize matter in the same direction.

$\vec{E}$ ⊖ ⊕ $\vec{p}$

2) $\mu > \mu_0$ or $\mu < \mu_0$ as we can have paramagnets and diamagnets

$\vec{B}$ $\vec{M}$ $\mu > \mu_0$ para mag. $\vec{B}$ $\vec{M}$ $\mu < \mu_0$ diamag.

when $\mu < \mu_0$ it's usually still quite close to μ_0 (parts per billion difference) [in fact $\mu \simeq \mu_0$ for many materials] so,

$$v = \frac{1}{\sqrt{\mu\varepsilon}} \equiv \frac{1}{n}\frac{1}{\sqrt{\mu_0\varepsilon_0}} = \frac{c}{n}$$ [where $n > 1$ for all known materials]

n ↳ index of refraction

- EM Waves can propagate inside Matter, but they are slower in matter (technically, in linear dielectrics)
- The wave equation is nearly identical ($\varepsilon_0 \to \varepsilon$; $\mu_0 \to \mu$) to the free space wave equation despite all the complicated physics!

For glass, $\varepsilon \cong 2.25\,\varepsilon_0$ so $n \simeq \sqrt{2.25} \simeq 1.5$
$\mu \simeq \mu_0$

For water, $\varepsilon \cong 80\,\varepsilon_0$ so $n \simeq \sqrt{80} \simeq 9$
$\mu \simeq \mu_0$ (But this is for static. Here at high frequencies we will find $n \neq 9$. More on this later!)

The physics here is kind of incredible!

the $\vec{E}$ & $\vec{B}$ are polarizing the matter, creating dipoles, which themselves produce time varying $\vec{E}$ & $\vec{B}$ fields, which superpose withe the incoming $\vec{E}$ & $\vec{B}$?

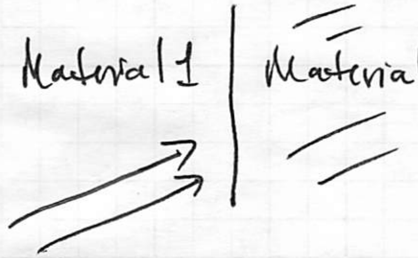
But it results in a simple wave with the same frequency it just moves more slowly!

So ω is the same, λ and v change.

What happens at the boundary between two materials?

ε is not homogeneous across the boundary it's changing.

→ Solve our wave equations in each medium and connect our solutions using Boundary Conditions.



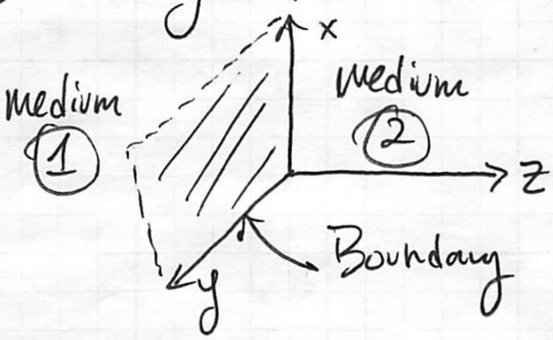
plane wave comes in. what happens?

This problem is the next "easiest" problem to solve after plane waves in homogeneous media (Important to Optics!)

Conventions (& Simplifications): we'll make a few choices to simplify our work, but our analysis is still fairly general.

1. Assume monochromatic (single ω) plane waves
   → other waves can be formed by summing monochromatic plane wave solutions. Highly idealized (∞ extent, etc.)

2. Usually make the boundary $z=0$ (i.e. the xy plane)

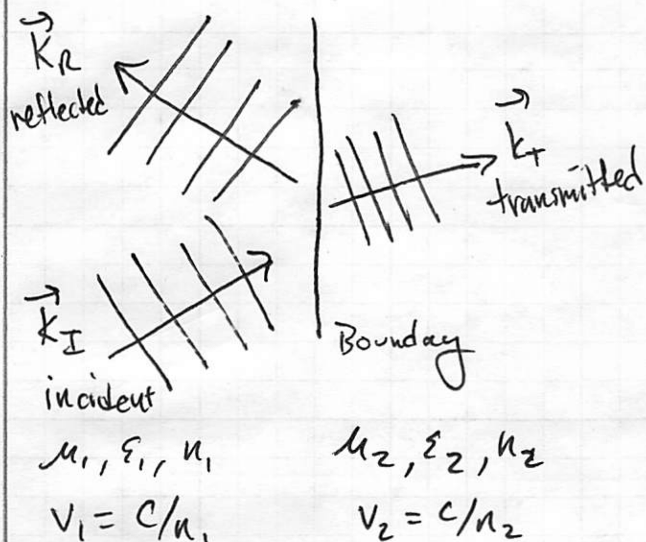


orientations can be changed or rotated (typically without loss of generality)

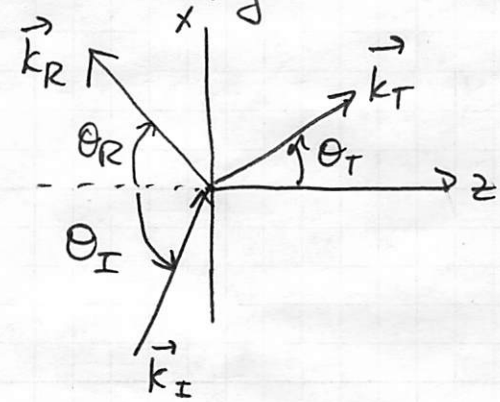
3. We usually assume incident waves come from $-z$, heading right. Then reflected wave superposes with incident in region 1 and transmitted is all you have in Region 2.

4. Regions 1 & 2 are each homogeneous & linear.

What this looks like for plane waves is shown below,



More simply



We will apply our Boundary Conditions at z=0 plane.

Claims (that we won't prove)

$$u = \frac{1}{2}\varepsilon E^2 + \frac{1}{2\mu}B^2 \qquad \vec{S} = \frac{1}{\mu}\vec{E}\times\vec{B}$$

$$I = \frac{1}{2}\varepsilon v E^2 \text{ (intensity)} \qquad \vec{B} = \vec{E}/v$$

Boundary Conditions (when $\vec{J}_f$ and ρ_f are zero!)

From $\nabla\cdot\vec{D} = \rho_f$ we get $\varepsilon_1 E_1^\perp = \varepsilon_2 E_2^\perp$

From $\nabla\cdot\vec{B} = 0$ we get $B_1^\perp = B_2^\perp$

From $\nabla\times\vec{E} = -\frac{d\vec{B}}{dt}$ we get $E_1^\parallel = E_2^\parallel$

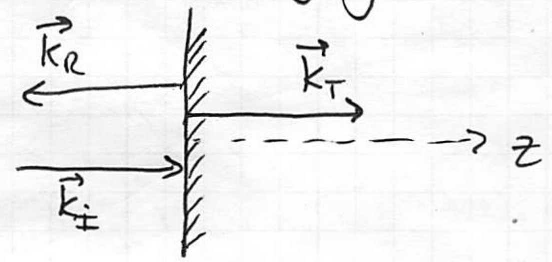
From $\nabla\times\vec{H} = \frac{d\vec{D}}{dt}$ we get $\frac{1}{\mu_1}B_1^\parallel = \frac{1}{\mu_2}B_2^\parallel$

We've seen these before. Do you remember how we got them? These are very general results!

These BCs tell us alot about how light behaves at interfaces and can help us understand; Eyeglasses, tele & microscopes, anti-reflective coatings etc., etc., etc.

Example: Normally Incident Light

We will start with simple light (monochromatic plane waves) impinging on a surface $\perp$ to that interface.



Incoming wave,

$$\tilde{\vec{E}}_I = \tilde{E}_{0I} e^{i(k_I z - \omega t)} \hat{n}$$

where $\hat{n}\cdot\hat{k} = 0$.

If $\vec{E}_I$ is linearly polarized, $\hat{n}$ is any constant vector (in xy plane)

So let's define our x axis to be the polarization axis.

(Still quite general, but makes our mathematics a touch easier.)

So we know in medium 1,

$$\tilde{\vec{E}}_I = \tilde{E}_{0I} e^{i(\vec{k}_I\cdot\vec{z} - \omega t)} \hat{x} \quad \text{and} \quad \tilde{\vec{B}}_I = \frac{\tilde{E}_{0I}}{v} e^{i(k_I z - \omega t)} \hat{y}$$

$\tilde{\vec{B}}_I$ must look like this (comes from Maxwell!)

Once this wave hits the boundary it will produce reflected & transmitted waves,

$$\tilde{\vec{E}}_T = \tilde{E}_{0T} e^{i(k_T z - \omega_T t)} \hat{n}_T \,; \quad \tilde{\vec{B}}_T = \hat{k}_T \times \tilde{\vec{E}}_T / v_2 \qquad v_2 = \frac{\omega_T}{k_T}$$

$$\tilde{\vec{E}}_R = \tilde{E}_{0R} e^{i(k_R z - \omega_R t)} \hat{n}_R \,; \quad \tilde{\vec{B}}_R = \hat{k}_R \times \tilde{\vec{E}}_R / v_1 \qquad \frac{\omega_I}{k_I} = \frac{\omega_R}{k_R} = v_1$$

These a ton of unknowns! amplitudes, frequencies, polarization!

→ You might guess that $\hat{n}_T = \hat{n}_R = \hat{x}$ (Why would polarization change?)

and that $\omega_I = \omega_R = \omega_T$ (Why would frequency change?)

The Boundary Conditions will tell us.

Boundary Condition I: $E_1^{\parallel} = E_2^{\parallel}$ (parallel to the boundary)

Because these waves are transverse — a normally incident wave will be automatically parallel to the boundary (direction of $\vec{E}$ is parallel)

So that means that $\vec{E}_1 = \vec{E}_2$ at the $z=0$ boundary!

$\vec{E}_1 = \vec{E}_I(z=0) + \vec{E}_R(z=0)$ and $\vec{E}_2 = \vec{E}_T(z=0)$

this gives us,

$$\tilde{E}_{0I}\, e^{+i(0-\omega_I t)}\,\hat{x} + \tilde{E}_{0R}\, e^{+i(0-\omega_R t)}\,\hat{n}_R = \tilde{E}_{0T}\, e^{+(0-\omega_T t)}\,\hat{n}_T$$

or

$$(\text{constant vector})\, e^{-i\omega_I t} + (\text{const. vec})\, e^{-i\omega_R t} = (\text{cons. vec})\, e^{-i\omega_T t}$$

$\Rightarrow$ There's no way for such a relationship to hold for all times unless $\omega_I = \omega_R = \omega_T$ (proved next page, but not here)

- this is reasonable, consider the simple case of $(\ )\cos(\omega_1 t) = (\ )\cos(\omega_2 t)$ for all $t \Rightarrow \omega_1 = \omega_2$!

- Boundaries will not change ω. Physics here is that waves cause oscillations in material at ω_I which produces waves at ω_I (reflected and transmitted waves)

$\Rightarrow$ same frequency but $v_1 \neq v_2$ so that k's and wavelengths are different in each media!

Proof: Suppose $Ae^{iat} + Be^{ibt} = Ce^{ict}$ for all t.

A, B, C are constant and non-zero. So we can divide everything by e^{iat},

$$A + Be^{i(b-a)t} = Ce^{i(c-a)t} \quad \text{for all } t.$$

At $t=0 \Rightarrow A+B=C$

At $t=\frac{2\pi}{b-a} \Rightarrow A+B = C e^{i2\pi\left(\frac{b-a}{c-a}\right)}$

these must be equal.

So it must be that $e^{i2\pi\left(\frac{b-a}{c-a}\right)} = 1$ and thus,

$\rightarrow b-a = c-a$ or $c=b$.

So if $b=c$, we can start again,

$$Ae^{iat} + Be^{ibt} = Ce^{ibt} \quad \text{for all } t$$

or

$$A = (C-B)\,e^{i(b-a)t} \quad \text{for all } t.$$

A is constant and nonzero, so $b=a$ so the time dependence vanishes!

Given that $\omega_I = \omega_R = \omega_T = \omega$ all the $e^{-i\omega t}$'s cancel so that,

$$\tilde{E}_{0I}\,\hat{x} + \tilde{E}_{0R}\,\hat{n}_R = \tilde{E}_{0T}\,\hat{n}_T \quad (\text{at } z=0) \qquad (\text{eqn. } 1)$$

Boundary Condition 2: $\frac{\vec{B}_1^{\parallel}}{\mu_1} = \frac{\vec{B}_2^{\parallel}}{\mu_2}$ at $z=0$

this says that,

$$\frac{\tilde{E}_{0I}}{\mu_1 v_1}\,\hat{y} + \frac{\tilde{E}_{0R}}{\mu_1 v_1}\left(-\hat{k}_I \times \hat{n}_R\right) = \frac{\tilde{E}_{0T}}{\mu_2 v_2}\left(\hat{k}_T \times \hat{n}_T\right) \qquad (\text{eqn. } 2)$$

Here we used $\vec{B} = \frac{\hat{k}\times\vec{E}}{v}$ and we noticed that $\vec{k}_R = -\vec{k}_I$ and cancelled all the $e^{-i\omega t}$'s!

If we assume that $\hat{n}_R$ and $\hat{n}_T$ can be anywhere in the xy plane,

$$\hat{n}_R = n_{Rx}\,\hat{x} + n_{Ry}\,\hat{y}$$
$$\hat{n}_T = n_{Tx}\,\hat{x} + n_{Ty}\,\hat{y}$$

when we work out the cross products and compare (1) + (2) we find them impossible unless $n_{Ry} = n_{Ry} = 0$

Summary: using the "parallel" component boundary conditions we learned all the ω's are the same and the polarization doesn't rotate.

[This is because of linearity → nonlinear materials can cause rotations.]

So from B.C. 1 $\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}$

and from B.C. 2 $\dfrac{\tilde{E}_{0I}}{\mu_1 v_1} - \dfrac{\tilde{E}_{0R}}{\mu_1 v_1} = \dfrac{\tilde{E}_{0T}}{\mu_2 v_2}$

With $\tilde{E}_{0I}$ given (we know the incident wave), $\tilde{E}_{0R}$ and $\tilde{E}_{0T}$ are unknown (2 eqn's and 2 unknowns).

Define $\beta \equiv \dfrac{\mu_1 v_1}{\mu_2 v_2}$ and thus,

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T} \quad \text{and} \quad \tilde{E}_{0I} - \tilde{E}_{0R} = \beta \tilde{E}_{0T}$$

* [Typically, $\mu_1 \simeq \mu_2 \simeq \mu_0$, so $\beta \simeq \dfrac{v_1}{v_2} = \dfrac{n_2}{n_1}$]

going from low n to high n (air to glass) $\beta > 1$

We can solve those 2 equations,

$$\tilde{E}_{0T} = \frac{2}{1+\beta} \tilde{E}_{0I} \left(\simeq \frac{2n_1}{n_1+n_2} \tilde{E}_{0I} \quad \text{if } \mu_1 \simeq \mu_2 \simeq \mu_0 \right)$$

$$\tilde{E}_{0R} = \frac{1-\beta}{1+\beta} \tilde{E}_{0I} \left(\simeq \frac{n_1 - n_2}{n_1+n_2} \tilde{E}_{0I} \quad \text{again if } \mu_1 \simeq \mu_2 \simeq \mu_0 \right)$$

These are the <u>Fresnel Equations</u>. Tell us about transmitted and reflected waves. Given $\tilde{E}_{0I}$ we know $\tilde{E}_{0R}$ and $\tilde{E}_{0T}$.

["parallel" B.C.s completely solved the problem.]

Notes: n's are <u>real</u>, no complex phases introduced

$n_2 > n_1$, $\tilde{E}_{0R}$ flips sign ($\tilde{E}_{0T}$ never flips)

$n_1 = n_2$ $\quad \tilde{E}_{0I} = \tilde{E}_{0T} + \tilde{E}_{0R} = 0$ (good!)

<u>What about the energy flow?</u>

Recall the intensity, $I = \frac{1}{2}\varepsilon v E_0^2$

We can define a transmission coefficient, which is the fraction of the incident intensity that is transmitted,

$$T \equiv \frac{I_T}{I_I} = \frac{\frac{1}{2}\varepsilon_2 v_2 E_{0T}^2}{\frac{1}{2}\varepsilon_1 v_1 E_{0I}^2} = \frac{\varepsilon_2 n_1}{\varepsilon_1 n_2}\frac{E_{0T}^2}{E_{0I}^2}$$

if we assume the magnetic properties of the materials are similar $\mu_1 = \mu_2 = \mu_0$ then,

$$n \equiv \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}} \cong \sqrt{\varepsilon/\varepsilon_0} \quad \text{so that} \quad \varepsilon_1 \simeq \varepsilon_0 n_1^2 \quad \varepsilon_2 \simeq \varepsilon_0 n_2^2$$

thus $T \cong \frac{n_2}{n_1}\frac{E_{0T}^2}{E_{0I}^2}$ given that $\tilde{E}_{0T} \simeq \frac{2n_1}{n_1+n_2}\tilde{E}_{0I}$,

$$T \simeq \frac{4n_1n_2}{(n_1+n_2)^2}$$

We can also define a reflection coefficient using a similar logic,

$$R \equiv \frac{I_R}{I_I} = \frac{E_{0R}^2}{E_{0I}^2} \cong \frac{(n_1-n_2)^2}{(n_1+n_2)^2}$$

Notes: $R+T=1$ (expression of conservation of energy!)

If $n_1 \simeq n_2$ $T \to 1$ and $R \to 0$ makes sense no changes

When $n_1 >> n_2$ or $n_2 >> n_1$ $T \to 0$ and $R \to 1$ (impedence mismatch)

Comments about R & T

if $n_1 >> n_2$, we have $\tilde{E}_{0T} \simeq 2\tilde{E}_{0I}$, $\tilde{E}_{0R} \simeq \tilde{E}_{0I}$ — This looks odd. We fully reflect, yet also transmit. Is this Ok?

heavy glass → air

Yes, because the energy flow (T & R) involve velocity, too (or n's).

As we saw, the much larger $v_2 \Longleftrightarrow$ much smaller n_2, makes $T \to 0$.

Energy is "bunched up" in the big n_1 side, it's stored in the polarization

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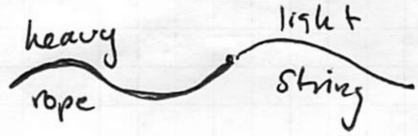
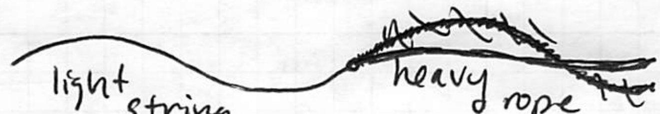
A simple analogy might help:



$n_1 >> n_2$

Yes the light string has an amplitude, it wiggles alot, but it isn't carrying away ~~much~~ of the energy (most of it reflects)

In the reverse case,



$n_1 << n_2$

Now $E_T \to 0$, $E_R \approx -E_I$ so @ $z=0$ no motion

again $T \to 0$ here.

In general mismatch at boundaries, there is poor transmission of energy.

Do not confuse w/ motion BTW