Wave Equation for Vector quantities

- We've seen the 1D wave equation: $\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$

  That gives rise to solutions: $f(x-vt)$

  We have sinusoidal solutions of the form:

  $$\tilde{f} = \tilde{A} e^{i(kx-\omega t)}$$

- We've seen the 3D wave equation: $\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$

  that gives rise to solutions: $f(\vec{r}-\vec{v}t)$

  We have sinusoidal solutions of the form:

  $$\tilde{f} = \tilde{A} e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

What about if the disturbance is a vector itself? (e.g. $\vec{E}$ or $\vec{B}$), then "f" is a vector quantity and our solutions will look like,

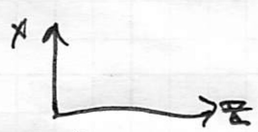
$$\vec{f}(\vec{r},t) = \underbrace{f_x(\vec{r},t)\hat{x} + f_y(\vec{r},t)\hat{y} + f_z(\vec{r},t)\hat{z}}_{\text{each of these is a wave}}$$

Our notation becomes a little nasty $\vec{f}_{physical}(\vec{r},t) = Re[\vec{\tilde{f}}(\vec{r},t)]$

Note: if $\vec{f}$ is $\perp$ to $\vec{k}$, the wave is "transversely polarized" or "transverse"

if $\vec{f}$ is $\parallel$ to $\vec{k}$, the wave is "longitudinally polarized" or "longitudinal"

For example, a wave moving in z-direction but vertically polarized (in x) is transverse



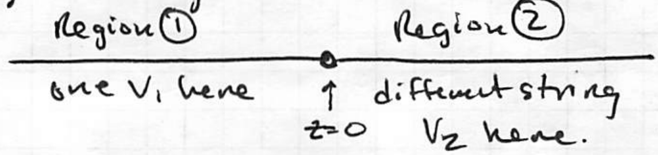
$$\vec{\tilde{f}}(z,t) = \tilde{A} e^{i(kz-\omega t)}\hat{x}$$

If $\vec{f}$ is transverse and $\vec{f}(\vec{r},t)$ always points in the same direction (as above), this wave is <u>linearly polarized</u>

Prelude to Waves & Boundaries

At a boundary between two media (strings, materials, etc.), a wave will typically be transmitted (into the new media) and reflected (into the old media).

In 1D you might imagine two different strings fused at a point.



An incident wave may come in from the left,

$$\tilde{f}_I(z,t) = \tilde{A}_I e^{i(k_1 z - \omega t)} \quad \text{with} \begin{cases} z<0 \text{ only} \\ k_1 > 0 \text{ (moves right)} \\ v_1 = \omega/k_1 \end{cases}$$

But, this will be superposed with the reflected wave, (left moving!)

$$\tilde{f}_R(z,t) = \tilde{A}_R e^{i(-k_1 z - \omega t)} \quad \text{with} \begin{cases} z<0 \text{ only} \\ k_1 > 0 \text{ still} \\ |v_1| = \omega/k_1 \text{ moves left.} \end{cases}$$

In Region 2, there will be a transmitted wave,

$$\tilde{f}_T(z,t) = \tilde{A}_T e^{i(k_2 z - \omega t)} \quad \text{with} \begin{cases} z>0 \\ k_2>0 \\ v_2 = \omega/k_2, \text{ right moving} \end{cases}$$

Claim: ω is the same for all these waves!

ω counts the wiggles/second and the boundary point has one wiggle frequency, ω.

Boundary Conditions will require that $f(z,t)$ & $\frac{\partial f}{\partial z}(z,t)$ be continuous. Soon we will find A_T and A_R given A_I

Electromagnetic Waves

Without Sources ($\rho = 0$; $\vec{J} = 0$), maxwell's equations are, (in vacuum)

$$\nabla \cdot \vec{E} = 0 \qquad \nabla \times \vec{E} = -d\vec{B}/dt$$
$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} = \mu_0 \varepsilon_0 \, d\vec{E}/dt$$

We will take the curl of Faraday's Law, we will find a 3D wave equation for $\vec{E}$,

$$\nabla \times (\nabla \times \vec{E}) = -\frac{d}{dt}(\nabla \times \vec{B})$$

$$\vec{\nabla}(\underbrace{\vec{\nabla} \cdot \vec{E}}_{0}) - \nabla^2 \vec{E} = -\frac{d}{dt}\left(\mu_0 \varepsilon_0 \frac{d\vec{E}}{dt}\right)$$

$$-\nabla^2 \vec{E} = -\mu_0 \varepsilon_0 \frac{d^2\vec{E}}{dt^2} \Rightarrow \boxed{\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{d^2\vec{E}}{dt^2}}$$

If we take $\nabla \times (\nabla \times \vec{B})$, we will get,

$$\boxed{\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \, d^2\vec{B}/dt^2}$$

That is both $\vec{E}$ & $\vec{B}$ satisfy 3D wave equations in vacuum.

In CARTESIAN COORDINATES for $\vec{E}$,

$$\nabla^2 \vec{E} \text{ means } \frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2}$$

So what we really have are <u>six</u>! equations (3 for $\vec{E}$ + 3 for $\vec{B}$)

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{dt^2} \equiv \frac{1}{v^2} \frac{d^2 E_x}{dt^2}$$

we have 5 more of these for

E_y, E_z, B_x, B_y, & B_z!

Each component of $\vec{E}$ & $\vec{B}$ satisfy a 3D wave equation with speed $v = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 2.998 \cdot 10^8 \, m/s$

- We've seen this before, we can have wave-like solutions as long as they travel at a speed $= c$ in vacuum!
- this suggests that light could be a wave but it's not yet proven here.

The solutions to the wave equations are very general and depend very much on initial conditions and boundary conditions.

We will limit our scope to the simplest, idealized solutions; plane wave solutions.

→ We can build up other solutions by superposing these solutions using the method of Fourier.

So we will investigate monochromatic plane wave solutions, (fixed ω)

$$\tilde{\vec{E}}(\vec{r},t) = \tilde{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

ω: frequency; $\omega = c|k|$

$\vec{k}$: wave vector; waves travels in $\vec{k}$ direction

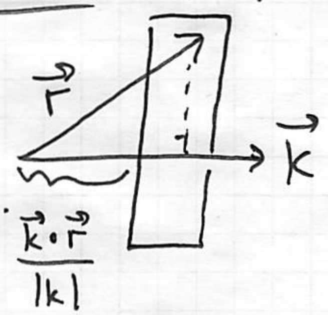
$\tilde{\vec{E}}_0$: amplitude vector

- It's constant!
- But, it has a phase $e^{i\delta}$

$\tilde{\vec{E}}$: Complex $\vec{E}$; the real physical $\vec{E}$ is given by $Re[\tilde{\vec{E}}]$.

How do we make plane waves?

Claim: If $\vec{k}\cdot\vec{r}$ = constant, that describes a plane



Any point on this plane will have the same value of $\vec{k}\cdot\vec{r}$
So that means $\tilde{E}$ is the same value for all points on this plane ($\perp$ to $\vec{k}$)

Lets assume $\vec{k}\cdot\vec{r}=0$, then we have a very simple plane wave, $\vec{\tilde{E}} = \vec{\tilde{E}}_0 e^{i\omega t}$

$\vec{\tilde{E}}_0$ fixed const.

$$\vec{\tilde{E}}_0 = \tilde{E}_{0x}\hat{x} + \tilde{E}_{0y}\hat{y} + \tilde{E}_{0z}\hat{z}$$

We've shown that Maxwell's Equations have wave solutions in free space, but if you pick some solution, you must check that the solution is consistent w/ all the Maxwell equations.

$\Rightarrow$ Doing this check will lead us to certain conditions on $\vec{E}$ & $\vec{B}$ (and $\vec{k}$!)

We will start this work by assuming a general plane wave solution.

$$\vec{\tilde{E}}(\vec{r},t) = \vec{\tilde{E}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

and then impose Maxwell's Equations on it.

Conditions on Plane Wave Solutions

start with $\tilde{\vec{E}}(\vec{r},t) = \tilde{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

Impose Gauss' Law $\nabla\cdot\vec{E}=0$,

$$\nabla\cdot\tilde{\vec{E}} = \nabla\cdot\tilde{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \frac{\partial}{\partial x}\tilde{E}_{0x} e^{i(k_x x+k_y y+k_z z-\omega t)}$$
$$+ \frac{\partial}{\partial y}\tilde{E}_{0y} e^{i(k_x x+k_y y+k_z z-\omega t)}$$
$$+ \frac{\partial}{\partial y}\tilde{E}_{0z} e^{i(k_x x+k_y y+k_z z-\omega t)}$$

$$\nabla\cdot\tilde{\vec{E}} = ik_x\tilde{E}_{0x} e^{i(\vec{k}\cdot\vec{r}-\omega t)} + ik_y\tilde{E}_{0y} e^{i(\vec{k}\cdot\vec{r}-\omega t)} + ik_z\tilde{E}_{0z} e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\nabla\cdot\tilde{\vec{E}} = i(\vec{k}\cdot\tilde{\vec{E}})e^{i(\vec{k}\cdot\vec{r}-\omega t)} = 0$$

if this is <u>always true</u> then $\vec{k}\cdot\tilde{\vec{E}}=0$!

This means that $\tilde{\vec{E}}$ must be perpendicular to $\vec{k}$, that is, the electric field is transversely polarized (in vacuum).

(Note: Fourier summing cannot change this!)

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if we assume that $\vec{B}$ has the same complex form, $\nabla\cdot\tilde{\vec{B}}=0$ tells us that $\vec{k}\cdot\tilde{\vec{B}}=0$ also so $\tilde{\vec{B}}$ must also be transverse!

Let's impose Faraday's Law, $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ should connect $\vec{E}$ & $\vec{B}$.

$$\nabla\times\vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \tilde{E}_x & \tilde{E}_y & \tilde{E}_z \end{vmatrix} = \hat{x}\left(\frac{\partial}{\partial y}\tilde{E}_z - \frac{\partial}{\partial z}\tilde{E}_y\right) - \hat{y}\left(\frac{\partial}{\partial x}\tilde{E}_z - \frac{\partial}{\partial z}\tilde{E}_x\right) + \hat{z}\left(\frac{\partial}{\partial x}\tilde{E}_y - \frac{\partial}{\partial y}\tilde{E}_x\right)$$

Given that $\tilde{E}_x = \tilde{E}_{0x} e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ and similar for $\tilde{E}_y + \tilde{E}_z$,

$$\nabla\times\vec{E} = \hat{x}\left(ik_y\tilde{E}_{0z} - ik_z\tilde{E}_{0y}\right)e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$
$$+\hat{y}\left(ik_x\tilde{E}_{0z} - ik_z\tilde{E}_{0x}\right)e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$
$$+\hat{z}\left(ik_x\tilde{E}_{0y} - ik_y\tilde{E}_{0x}\right)e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$
$$= i(\vec{k}\times\vec{\tilde{E}}_0)e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

(the arguments in parentheses are components of this crossproduct!)

$$-\frac{d\vec{B}}{dt} = -\frac{d}{dt}\vec{\tilde{B}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = i\omega\vec{\tilde{B}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

so Faraday's Law suggests, $\nabla\times\vec{E} = -\frac{d\vec{B}}{dt}$,

$$i(\vec{k}\times\vec{\tilde{E}}_0)e^{i(\vec{k}\cdot\vec{r}-\omega t)} = i\omega\vec{\tilde{B}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

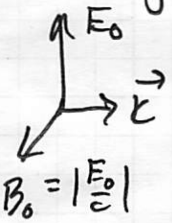
Thus we find that $\vec{k}\times\vec{\tilde{E}}_0 = \omega\vec{\tilde{B}}_0$. So $\vec{E}$ & $\vec{B}$ are transverse and perpendicular to eachother!

For example, if $\vec{k} \Rightarrow \hat{z}$ direction,

$$\vec{\tilde{B}}_0 = \frac{k}{\omega}(\hat{z}\times\vec{\tilde{E}}_0) = \frac{1}{c}(\hat{z}\times\vec{\tilde{E}}_0)$$

→ Both $\vec{E}$ + $\vec{B}$ are transverse; $\vec{B}$ & $\vec{E}$ are perpendicular to eachother and $|B_0| = \frac{1}{c}|E_0|$

→ Finally $\tilde{B}_0$ is in phase with $\tilde{E}_0$ (no complex issues!)



General traveling wave description

Note: we relied on complex analysis to simplify things.

$$\nabla\times\vec{\tilde{A}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = i\vec{k}\times\vec{\tilde{A}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$
$$\nabla\cdot\vec{\tilde{A}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = i\vec{k}\cdot\vec{\tilde{A}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$
$$\frac{d}{dt}\vec{\tilde{A}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = -i\omega\vec{\tilde{A}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

There's one last Maxwell Equation, but it doesn't give any more constraints (it's redundant)

So our full plane wave solutions that obey all 4 Maxwell Equations in vacuum are:

$\vec{\tilde{E}}(\vec{r},t) = \vec{\tilde{E}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} \hat{n}$ $\quad$ $\hat{n}$ is the polarization direction of $\vec{\tilde{E}}_0$, $\hat{n}\cdot\hat{k}=0$

$$\vec{\tilde{B}}(\vec{r},t) = \vec{\tilde{B}}_0 e^{+i(\vec{k}\cdot\vec{r}-\omega t)}(\hat{k}\times\hat{n}) = \frac{\vec{\tilde{E}}_0}{c} e^{i(\vec{k}\cdot\vec{r}-\omega t)}(\hat{k}\times\hat{n}) = \frac{\hat{k}\times\vec{\tilde{E}}}{c}$$

Remember that the physical $\vec{E}$ is,

$$\vec{E}_{physical}(\vec{r},t) = Re[\vec{\tilde{E}}] = \vec{E}_0\cos(\vec{k}\cdot\vec{r}-\omega t+\delta)$$

(the δ is hidden in the $\vec{\tilde{E}}_0$)

$$\vec{B}_{physical} = \frac{\hat{k}\times\vec{E}_{physical}}{c}$$

We've learned that electric & magnetic fields carry energy & momentum – let's look at waves to see how they carry both.

Energy & Momentum in Waves

Consider our plane wave:

$$\vec{E}(\vec{r},t) = \text{Re}\left[\tilde{\vec{E}}(\vec{r},t)\right] = \vec{E}_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \delta)$$

Let's assume its linearly polarized: $\vec{E}_0 = E_0 \hat{n}$

where $\hat{n}$ is fixed and $\perp$ to $\vec{k}$.

Also, $\vec{B} = \hat{k} \times \vec{E}/c$ (and since $\vec{E} \perp \hat{k}$, $|B_0| = |E_0|/c$)

→ the energy density in an electromagnetic field is,

$$u_{em} = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2$$

→ For the plane wave where $|B_0| = |E_0|/c$,

$$u_{em}^{P.W.} = \frac{1}{2}\varepsilon_0 |E_0|^2 \cos^2(\vec{k}\cdot\vec{r} - \omega t + \delta) + \frac{1}{2\mu_0}\frac{|E_0|^2}{c^2}\cos^2(\vec{k}\cdot\vec{r} - \omega t + \delta)$$

Because $\frac{1}{\mu_0 \varepsilon_0} = c^2$, $\frac{1}{2\mu_0 c^2} = \frac{1}{2}\varepsilon_0$ so both the terms are identical.

In plane waves $\vec{E}$ & $\vec{B}$ are symmetric in the sense that at every point, they store equal amounts of energy!

$$u_{em}^{P.W.} = \varepsilon_0 |E_0|^2 \cos^2(\vec{k}\cdot\vec{r} - \omega t + \delta) \quad \left\{\begin{array}{c}\text{scalar quantity} \\ \text{that varies in time} \\ \text{and space, but is} \\ \text{always } \geq 0\end{array}\right\}$$

Time average (at any/every pt.),

$$\langle u_{em} \rangle = \frac{1}{T}\int_0^T u_{em}(\vec{r},t)\,dt = \frac{1}{2}\varepsilon_0 E_0^2 \quad \text{b/c} \quad \langle \cos^2(\vec{k}\cdot\vec{r} - \omega t + \delta)\rangle = \frac{1}{2}$$

Warning! In complex notation, if you try to compute a complex energy density, you must be careful,

$$\tilde{u}_{em} \overset{?}{=} \frac{1}{2}\varepsilon_0 \tilde{E}_0^2 e^{2i(\vec{k}\cdot\vec{r}-\omega t+\delta)} + \frac{1}{2\mu_0}\tilde{B}_0^2 e^{2i(\vec{k}\cdot\vec{r}-\omega t+\delta)}$$

This fails $\underline{\mathrm{Re}(z^2) \neq \mathrm{Re}(z)^2}$

Taking $\mathrm{Re}[\tilde{u}_{em}]$ gives $\cos(2(\vec{k}\cdot\vec{r}-\omega t+\delta))$, which is not right! $\mathrm{Re}(\tilde{u}_{em}) \neq u_{em}$!

Poynting Vector $\underline{\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}}$

- You must be careful using complex notation. That works for linear operations only! it fails for quadratic ones that show up when computing u or $\vec{S}$.

no! $\times$ $\tilde{\vec{S}} \neq \frac{1}{\mu_0}\tilde{\vec{E}}\times\tilde{\vec{B}}$ that doesn't work, don't do it.

To find $\vec{S}$, use the true $\vec{E}$ & $\vec{B}$ from before,

$$\vec{S} = \frac{1}{\mu_0}\mathrm{Re}[\tilde{\vec{E}}]\times \mathrm{Re}[\tilde{\vec{B}}] = \frac{1}{\mu_0}[\vec{E}_0\times\vec{B}_0]\cos^2(\vec{k}\cdot\vec{r}-\omega t+\delta)$$

What direction is that? $\vec{E}_0\times\vec{B}_0$?

$$\vec{E}_0\times\vec{B}_0 = \vec{E}_0\times\left(\hat{k}\times\frac{\vec{E}_0}{c}\right) \underset{\text{product rule \#2}}{=} \hat{k}\left(\vec{E}_0\cdot\frac{\vec{E}_0}{c}\right) - \frac{\vec{E}_0}{c}\underbrace{(\vec{E}_0\cdot\hat{k})}_{0 \;\; \vec{E}_0\perp\hat{k}}$$

$$\vec{E}_0\times\vec{B}_0 = \hat{k}\left(\frac{E_0^2}{c}\right)$$

Thus, $\vec{S} = \frac{E_0^2}{\mu_0 c}\cos^2(\vec{k}\cdot\vec{r}-\omega t+\delta)\hat{k}$ with $c^2 = \frac{1}{\mu_0\varepsilon_0}$

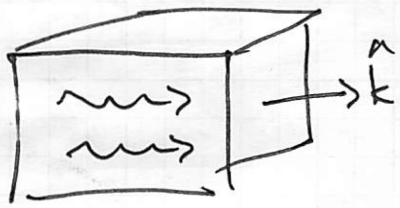
Coefficient out front: $\frac{E_0^2}{\mu_0 c} = \varepsilon_0 c E_0^2 = \sqrt{\frac{\varepsilon_0}{\mu_0}}E_0^2$ all are ok!

Energy flows in the direction of propagation! $\hat{k}$.

With $u_{em} = \varepsilon_0 E_0^2 \cos^2(\vec{k}\cdot\vec{r} - \omega t + \delta)$,

$\vec{S} = u_{em} c \hat{k}$ energy is transported at the speed of light in vacuum.

Consider a cube that is 1 light second deep,



In 1 second, all the EM energy will exit the volume!

$\rightarrow$ 1 light sec. $\leftarrow$

The time average of $\vec{S}$ at any point is,

$$\langle \vec{S} \rangle = \underbrace{\tfrac{1}{2}}_{\langle \cos^2 \rangle} \varepsilon_0 E_0^2 c \hat{k} = \langle u_{em} \rangle c \hat{k}$$

So that's the energy story, what about the momentum density, $\vec{P}_{EM}$?

<u>Momentum Density</u> $\vec{P}_{EM}$ –

Recall we defined $\vec{P}_{em}$ in terms of $\vec{S}$,

$$\vec{P}_{em} = \mu_0 \varepsilon_0 \vec{S} \quad \text{so at any point,}$$

$$\vec{P}_{EM} = \mu_0 \varepsilon_0 \frac{E_0^2}{\mu_0 c} \cos^2(\vec{k}\cdot\vec{r} - \omega t + \delta)\hat{k} = \frac{\varepsilon_0 E_0^2}{c} \cos^2(\vec{k}\cdot\vec{r} - \omega t + \delta)\hat{k}$$

It also points in the direction of propagation! $\hat{k}$

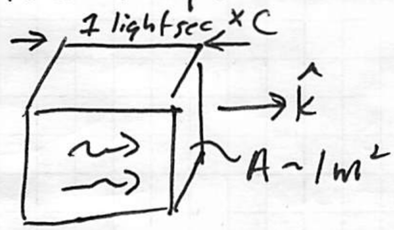
$$\langle \vec{P}_{em} \rangle = \mu_0 \varepsilon_0 \langle u_{em} \rangle c \hat{k} = \frac{\langle u_{em} \rangle}{c} \hat{k} =$$

$$\langle \vec{P}_{em} \rangle = \frac{\varepsilon_0 E_0^2}{2c} \hat{k}$$

hmm... this makes me think about $p = E/c$ from Phy 215!
(more on this later!)

<u>Observation</u>: If light hits matter and is absorbed, there will be a force.

Consider the 1 light second box,



1 light second deep and $1m^2$ cross section

The total momentum stored in the box is,

$$\vec{p} = \vec{\mathbb{P}}_{em} \times \text{volume} = \underbrace{\frac{\langle u_{em}\rangle \hat{k}}{c}}_{\langle \vec{\mathbb{P}}_{em}\rangle} \cdot \underbrace{c\,(1\,\text{sec})(1m^2)}_{\text{volume}}$$

in 1 second all this momentum flows out of the box, and gets absorbed by the right wall.

So the average momentum change per second is,

$$\underbrace{\frac{\Delta \vec{P}_{absorbed}}{1\,\text{sec}}}_{} = \frac{\langle u_{em}\rangle}{c} \cdot c\,\frac{1\,\text{sec}\ 1m^2}{1\,\text{sec}}\hat{k} = \langle u_{em}\rangle 1m^2\,\hat{k}$$

↑ this is a force due to radiation $\vec{F}$

$$\langle \mathbb{P}_{em}\rangle \cdot \frac{V_{box}}{time} \text{ (direction)}$$

We can concieve of this average force per unit area, as a pressure,

$P_{radiation} = \langle u_{em}\rangle$

$F/A \equiv$ radiation pressure $= \langle u_{em}\rangle$

(Note: If it were reflected $\Delta\vec{p}$ would be twice as high and so would the radiation pressure)

Finally, we will often be interested in the average power transported per unit area,

$$\frac{\langle \text{Power}\rangle}{\text{unit area}} \equiv \text{Intensity} \equiv I \quad (\text{this is } \langle\vec{S}\rangle\cdot\hat{k})$$

$$I = \langle|\vec{S}|\rangle = \langle u_{em}\rangle c = \tfrac{1}{2}\varepsilon_0 c E_0^2 = P_{radiation}\cdot c$$