Through a variety of experimental work and theoretical development in the 1800's led to the following set of equations,

$$\nabla\cdot\vec{E} = \rho/\epsilon_0 \quad \text{Gauss}$$
$$\nabla\times\vec{E} = -d\vec{B}/dt \quad \text{Faraday}$$
$$\nabla\cdot\vec{B} = 0$$
$$\nabla\times\vec{B} = \mu_0\vec{J} \quad \text{Ampere}$$

$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$ Lorentz force on charges.

- Maxwell had these equations and models in the 1860's. He didn't invent these equations, he <u>started</u> here.
- Maxwell (and others) asked, "Could these be a complete theory of electromagnetism?"
- We've used them for time dependent work and computed a number of experimentally verifiable results. (so maybe it is?)

<u>Is it complete</u>?

- Let's do some mathematical manipulations to see what we find (this is similar to what Maxwell did).

Fact: the divergence of any curl is zero.

- This is provable, mathematical result. $\nabla\cdot(\nabla\times\vec{G}) = 0$

Let's check this with Faraday's Law, $\nabla\times\vec{E} = -d\vec{B}/dt$

the mathematics tells us $\nabla\cdot(\nabla\times\vec{E}) = 0$

so it must be that $\nabla\cdot\left(-\frac{d\vec{B}}{dt}\right) = 0$

and it is! $\nabla\cdot\left(-\frac{d\vec{B}}{dt}\right) = -\frac{d}{dt}(\nabla\cdot\vec{B}) = 0$ (math & physics agree!)

Well, what about Ampere's Law?

- the mathematics says $\nabla\cdot(\nabla\times\vec{B})=0$, but does this agree with the physics?

With $\nabla\times\vec{B}=\mu_0\vec{J}$, we have $\nabla\cdot(\mu_0\vec{J})=0$

- that's not <u>always</u> true. Sometimes $\nabla\cdot\vec{J}=0$, but usually is $\nabla\cdot\vec{J}=-d\rho/dt$ (conservation of charge)

So we have an incomplete theory. Maxwell realized that the set of equations cannot be the whole story, because they give $\nabla\cdot\vec{J}=0$ always, which is physically <u>not</u> the case sometimes.

<u>When is this problematic?</u>

If $\nabla\cdot\vec{J}=0$ (always...) then $\iiint_V \nabla\cdot\vec{J}\,d\tau=0 \Rightarrow \oiint_{\text{Any surface}} \vec{J}\cdot d\vec{A}=0$

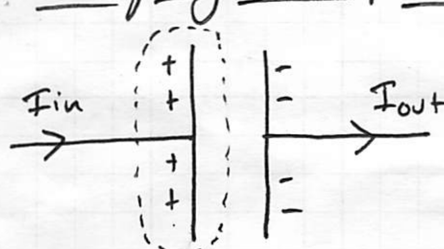
With $\vec{J}$ as the current density,

$\vec{J}\cdot d\vec{A}$ tells you the current leaving.

that means $\oiint\vec{J}\cdot d\vec{A}=0$ says the current in = current out <u>always</u> in <u>all</u> circumstances.

So we can easily come up with a case where this is a problem.

<u>Charging a Capacitor</u>



As the charge builds up ($Q(t)$), $\oiint\vec{J}\cdot d\vec{A}\neq 0$ current goes in, but none comes out for the dashed region.

In this case,

$$\oiint \vec{J}\cdot d\vec{A} = \begin{matrix}\text{net outflow}\\ \text{of current}\end{matrix} = -\frac{d}{dt} Q_{\text{enclosed}}$$

This is physically just conservation of charge,

$$\nabla\cdot\vec{J} = -d\rho/dt$$

Ampere's Law gave us $\mu_0 \nabla\cdot\vec{J} = 0$, which is wrong in general.

So the completing of this theory means examining Ampere's Law a bit more closely.

The problem we have is when $d\rho/dt \neq 0$ anywhere.

So electro- and magnetostatics are fine (glued charges or steady currents).

But a true electrodynamic theory must solve this $\nabla\cdot\vec{J}$, $d\rho/dt$ issue!

How did Maxwell fix this problem?

- Let's keep going with the capacitor example.

What's the magnetic field near this loop?

Ampere's Law says $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} = \mu_0 \iint \vec{J}\cdot d\vec{A}$

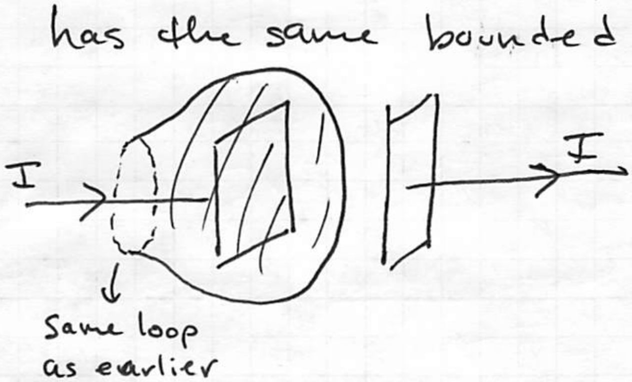
any surface bounded by the loop.

It looks like we should get $\vec{B}_{\text{long wire}}$ ...

But Ampere's Law is true for any surface that is bounded by the loop.

Up to now, this hasn't been a problem, but we can see how it becomes a problem here.

think about the "soap bubble" surface that has the same bounded loop,



This new surface isn't a flat circle, but is bounded by the same loop as the flat circle.

for this surface $I_{thru} = 0$, no current pokes through the soap bubble surface!

Ampere's Law (as written) is failing here.

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \underbrace{\iint \vec{J}\cdot d\vec{A}}_{\text{any surface bounded by loop}}$$

} we came up with 2 surfaces that give different results for the right-hand side

the problem is that $\frac{d\rho}{dt} \neq 0$ inside the bubble.
→ charge builds up so that current in ≠ current out.

(If we had steady currents, like in the past, and $d\rho/dt = 0$, we'd have no issue, like in the past.)

Maxwell's Correction

$\nabla\cdot(\nabla\times\vec{B}) = 0$ must be true! the math is undeniable.

If $\nabla\times\vec{B} = \mu_0\vec{J}$, we get $\nabla\cdot(\mu_0\vec{J}) = 0$ instead of what conservation of current says,

$$\nabla\cdot(\mu_0\vec{J}) = -\mu_0\, d\rho/dt$$

Let's see if we can fix Ampere's Law. Add $\vec{X}$ to $\mu_0\vec{J}$.

$\nabla\times\vec{B} = \mu_0\vec{J} + \vec{X}$ let's see what happens with this.

With $\nabla \times \vec{B} = \mu_0 \vec{J} + \vec{X}$,

$$\nabla \cdot (\nabla \times \vec{B}) = \mu_0 \nabla \cdot \vec{J} + \nabla \cdot \vec{X}$$

$$\underset{\text{from the math}}{0} = \underbrace{-\mu_0 \, d\rho/dt}_{\substack{\text{conservation} \\ \text{of current.}}} + \nabla \cdot \vec{X}$$

Maybe it's still not obvious what $\vec{X}$ is, but let's push forward. The form suggests we peek into Gauss' Law,

$$\nabla \cdot \vec{E} = \rho/\varepsilon_0 \Rightarrow \rho = \varepsilon_0 \nabla \cdot \vec{E}$$

thus,

$$\frac{d\rho}{dt} = \varepsilon_0 \frac{d}{dt}(\nabla \cdot \vec{E}) = \varepsilon_0 \nabla \cdot \left(\frac{d\vec{E}}{dt}\right)$$

Oh! it looks like if $\vec{X} \equiv +\mu_0 \varepsilon_0 \frac{d\vec{E}}{dt}$,

then the equation is always satisfied!

$$\boxed{\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \, d\vec{E}/dt}$$

Ampere's Law w/ Maxwell Correction

This extra term is the "displacement current", $\mu_0 \vec{J}_D$ with $\vec{J}_D = \varepsilon_0 \, d\vec{E}/dt$

→ It has units of current density, but it's not a physical flow of charge. It's not a current. The name was Maxwell's but now it's just what we have.

→ Note: In statics with $d\vec{E}/dt = 0$, we are back to the old Ampere's Law,

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

Why didn't Ampere (or others) notice this?

→ the effect is very small unless $d\vec{E}/dt$ is big.

→ In SI units $\varepsilon_0 \mu_0 = \frac{1}{9 \cdot 10^{16}}\ s^2/m^2$ tiny #'s!

the correction term was undetectable in those days in the experiments that were being conducted.

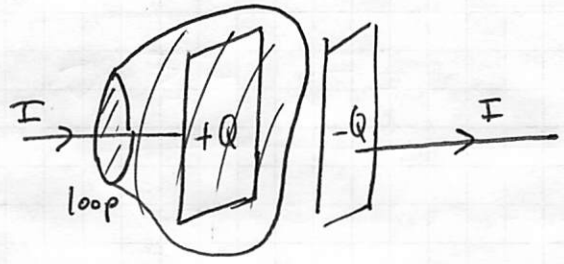
Only when $d\vec{E}/dt$ is huge does it matter, usually $\vec{J}$ will dominate!

We have very nice symmetry now!

→ Changing $\vec{B}$ creates a curly $\vec{E}$ (Faraday)

→ Changing $\vec{E}$ creates a curly $\vec{B}$ (Maxwell correction)

How do we resolve the case we proposed?



Inside the capacitor,

$$E \simeq \frac{Q}{A\varepsilon_0}$$

This gives us, $\mu_0 \varepsilon_0 \frac{dE}{dt} = \frac{\mu_0}{A}\frac{dQ}{dt} = \frac{\mu_0}{A} I_{in}$

so with the fixed up Ampere's Law,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{through} + \mu_0 \varepsilon_0 \iint_{Area} \frac{d\vec{E}}{dt} \cdot d\vec{A}$$

When the area is the flat loop, $E \simeq 0$ so the right hand side is just $\mu_0 I$.

when we have the bubble $I_{thru} = 0$ but,

$$\iint \frac{\mu_0 I}{A}\, dA = \mu_0 I \quad \text{same result!}$$

Going backwards from the new Ampere's law,

$0 = \nabla \cdot (\nabla \times \vec{B})$ is a pure mathematical fact.

Combined with Gauss's law $\nabla \cdot \vec{E} = \rho/\epsilon_0$ we recover conservation of charge!

$$\nabla \cdot \vec{J} = -d\rho/dt$$

In other words, Maxwell's Equations imply/require/give charge conservation!

→ We don't tack this on as an extra fact of nature, it's built into Maxwell's Eqns.

→ And (as we will see) so is relativistic invariance.

The set of Eqn's is now complete. It is a full, self consistent field theory. And as postulated, leads to more conservation theorems (energy & momentum)

<u>This is the greatest synthesis in physics!</u>

$$\boxed{\begin{aligned} \nabla \cdot \vec{E} &= \rho/\epsilon_0 \\ \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{E} &= -\frac{d\vec{B}}{dt} \\ \nabla \times \vec{B} &= \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{d\vec{E}}{dt} \end{aligned}}$$

One last thing, that we will come back to ....
We learned a lot by taking $\nabla\cdot$ (Maxwell curl eqns)
What about taking $\nabla\times$?

$$\nabla\times(\nabla\times\vec{E}) = \nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E}$$

in empty space $\rho = 0$ so that $\nabla\cdot\vec{E} = 0$ and thus,

$$\nabla\times(\nabla\times\vec{E}) = -\nabla^2\vec{E}$$

$\nabla\times\vec{E} = -d\vec{B}/dt$ so that,

$$\nabla\times(\nabla\times\vec{E}) = -\nabla\times\left(\frac{d\vec{B}}{dt}\right) = -\frac{d}{dt}(\nabla\times\vec{B}) \quad \text{and.}$$

$\nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\dfrac{d\vec{E}}{dt}$ in empty space, $\vec{J} = 0$
so that,

$$\nabla\times\vec{B} = \mu_0\varepsilon_0\frac{d\vec{E}}{dt} \quad \text{and,} \quad -\frac{d}{dt}(\nabla\times\vec{B}) = -\mu_0\varepsilon_0\frac{d^2\vec{E}}{dt}$$

Combining all of this we find,

$\nabla\times(\nabla\times\vec{E}) = -\dfrac{d}{dt}(\nabla\times\vec{B})$ just gives,

$$-\nabla^2\vec{E} = -\mu_0\varepsilon_0\frac{d^2\vec{E}}{dt^2}$$

(we get the same for $\vec{B}$ using $\nabla\times(\nabla\times\vec{B})$)

$$\hookrightarrow \nabla^2\vec{E} = \mu_0\varepsilon_0\frac{d^2\vec{E}}{dt^2}$$

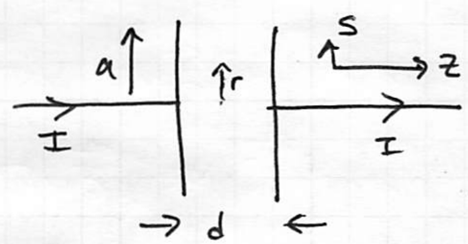
(1 component $\nabla^2 E_z = \mu_0\varepsilon_0\dfrac{d^2E_z}{dt^2}$)

You might have seen this before,
the 1 dimensional wave equation $\rightarrow \dfrac{\partial^2 f}{\partial x^2} = \dfrac{1}{v^2}\dfrac{\partial^2 f}{\partial t^2}$
(solutions are travelling waves with speed v).

So in empty space we will find solutions for $\vec{E}$ & $\vec{B}$ that are not zero! They will be travelling waves with $v = \dfrac{1}{\sqrt{\varepsilon_0\mu_0}} = 3\cdot10^8\,m/s$!

Example: Charging Up a Capacitor

Consider a large parallel plate capacitor made of two metal circular plates (radius, a) separated by a distance d ($d \ll a$). Current runs through the circuit charging the plates. On the positive plate the charge increases $Q(t) = Q_0 + \beta t$.
[the linear relationship here is just a model not always true as it depends on $I(t)$]



We aim to determine the magnetic field produced by the changing electric field between the plates.
(we will neglect fringe effects)

The electric field in the plates increases as Q does,

$$\vec{E} = \frac{Q(t)}{A\varepsilon_0}\hat{z}$$

What is the direction of $\vec{B}$?

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \varepsilon_0 \mu_0 \frac{d\vec{E}}{dt}$$

$\vec{J} = 0$,

$$\nabla \times \vec{B} = \varepsilon_0 \mu_0 \frac{d\vec{E}}{dt}$$

$$\iint_S (\nabla \times \vec{B}) \cdot d\vec{a} = \oint_C \vec{B} \cdot d\vec{l} = \varepsilon_0 \mu_0 \iint_S \frac{d\vec{E}}{dt} \cdot d\vec{a}$$

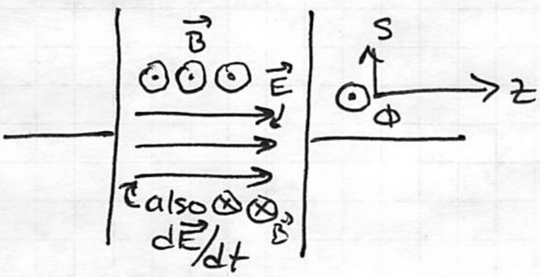
If we go back to thinking about Ampere's Law, we can see $\vec{B}$ circulates around $d\vec{E}/dt$ (like it does w/ $\vec{J}$.)

$$\oint_C \vec{B}\cdot d\vec{l} = \iint_S \mu_0 \varepsilon_0 \frac{d\vec{E}}{dt}\cdot d\vec{A}$$

From this, we ~~ ~~ expect that magnetic field to curl around $d\vec{E}/dt$.

As the direction of $\vec{E}$ remains unchanged, only its magnitude increases in this case, $\vec{E}(t) = \frac{Q(t)}{A\varepsilon_0}\hat{z}$, the magnetic field circulates around the electric field. $\pm\hat{\phi}$.

Which direction does it circulate? $+\hat{\phi}$ or $-\hat{\phi}$?



b/c Q(t) increasing.

Use the right hand rule, like we did with $\vec{J}$.
Careful: b/c $d\vec{E}/dt$ matters not $\vec{E}$!

So $\vec{B}$ points in the $+\hat{\phi}$ direction inside the capacitor plates.

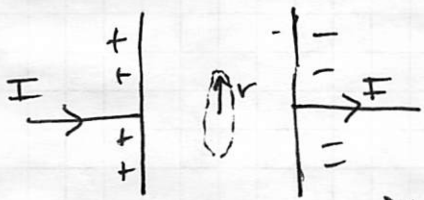
How do we calculate $\vec{B}$?

Because $\vec{E}$ changes the same way everywhere in the capacitor region, we expect $\vec{B}$ to be translationally invariant in z. also, we expect that rotating the system in ϕ has no effect $\rightarrow$ azimuthal symmetry so ϕ can't matter either. Thus,

$$\vec{B}(s,\phi,z) = B(s)\hat{\phi}$$

This helps us choose a loop to use.

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 \varepsilon_0 \iint_S d\vec{E}/dt \cdot d\vec{A}$$



we choose a circular loop that is plane parallel to the plates. $d\vec{A} = s\,d\phi\,ds\,\hat{z}$

with $\vec{E}(t) = \dfrac{Q(t)}{A\varepsilon_0}\hat{z}$, $\dfrac{d\vec{E}}{dt} = \dfrac{dQ/dt}{A\,\varepsilon_0}\hat{z}$

so that $(Q(t) = Q_0 + \beta t)$,

$$\frac{d\vec{E}}{dt} = \frac{\beta}{A\varepsilon_0}\hat{z}$$

thus,

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \varepsilon_0 \iint_S \frac{\beta}{A\varepsilon_0}\hat{z}\, s\,d\phi\,ds\,\hat{z}$$

$$B 2\pi r = \frac{\mu_0 \varepsilon_0 \beta}{A\varepsilon_0}(\pi r^2)$$

so $\vec{B} = \dfrac{\mu_0 \beta}{\pi a^2}\dfrac{r}{2}\hat{\phi}$ check units & limits

$$[B] = [T] = \frac{[\mu_0][\beta][r]}{[a^2]} = \frac{[N/A^2][A][m]}{[m^2]}$$

$$= \frac{N}{mA} = T \checkmark$$

as $r \to 0$, $\vec{B} \to 0$ no enclosed $\vec{E}$ flux $\checkmark$