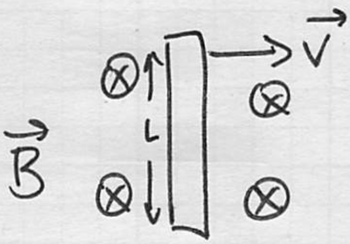
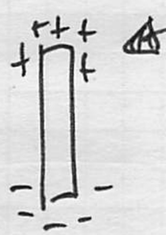
Motional EMF

Motional EMF is a mechanism that generates an EMF through motion. It's very common; its how generators work! It will also help us get to Faraday's discoveries!

Consider a metal bar moving in a uniform magnetic field.

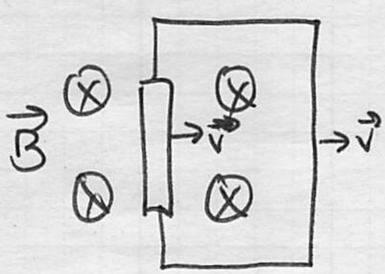


Charges inside the bar feel a force $= \vec{v} \times \vec{B}$ (up for + charges in this case)



$\vec{f} = \vec{v} \times \vec{B}$ causes a separation of charge, which creates an EMF. This bar can be connected in a circuit.

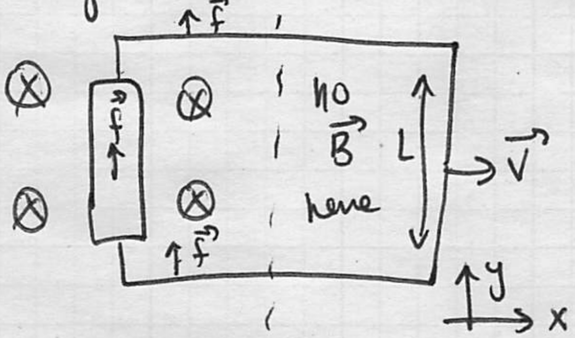
If your bar is connected to a wire, then we could have a circuit.



If $\vec{B}$ is uniform and the whole setup stay inside the field, then nothing happens.

$\vec{f}$ points up on both the left + right legs. so $\oint \vec{f} \cdot d\vec{l} = 0$ the edges cancel. NO EMF.

If you are leaving the field then the right leg no longer cancels,



Now,

$$\mathcal{E} = \oint \vec{f}\cdot d\vec{l}$$

$$= vBL + \underbrace{0 + 0}_{\substack{\text{top/bottom} \\ \vec{f} \perp d\vec{l}}} + \underbrace{0}_{\substack{B=0 \\ \text{on right}}}$$

We've built up a static charge difference that creates an $\mathcal{E}MF = \Delta V_{open\ end} = vBL$

If we attach a resistor to the circuit, R, then the current is $I = \frac{EMF}{R} = \frac{vBL}{R}$

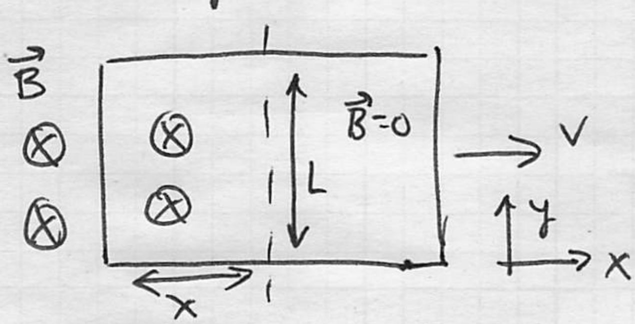
The line integral $\oint \vec{f}\cdot d\vec{l}$ can be very hard to compute if the shape of the loop is complicated or if the field is complicated.

But there's a little feature of the math that can get us out of trouble.

Changing Flux

Let's think about the magnetic flux through the loop.

$$\Phi_B = \iint \vec{B}\cdot d\vec{a}$$



$$\Phi_B = BL|x|$$

Let's take the time derivative of Φ_B

$$\frac{d\Phi_B}{dt} = BL\frac{d|x|}{dt} = -BLv$$

(the minus sign comes in b/c $|x|$ gets smaller!)

So for this particular case, we find that:

$$\mathcal{E} = EMF = BLV = -d\Phi_B/dt$$

We can prove, but we won't here (Griffiths 7.1.3), that the flux relationship is always true!

so

$$\underline{\mathcal{E} = -\frac{d\Phi_B}{dt}}$$

gives us the EMF for any loop shape, no matter how complex $\vec{B}$ is.

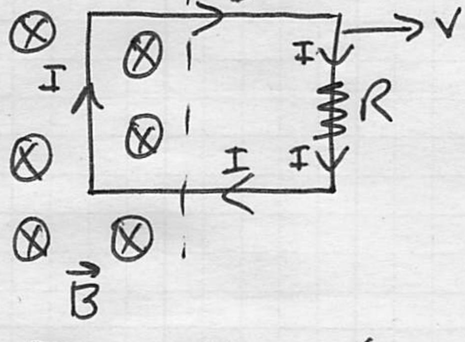
- Solid objects moving in these situations experience EMFs and currents can be driven in these bulk materials. These are called "Eddy Currents."

  Eddy Currents are currents that drift around the materials w/o a battery.

- Do these things violate energy conservation?!

  Let's see how they don't.

Consider our canonical setup,



in this setup, we generate an $EMF = \mathcal{E} = BLV$

the current we see is

$$I = \frac{\mathcal{E}}{R} = \frac{BLV}{R}$$

But there's a current running through the left branch, which experiences a magnetic force to the ~~right~~ left (pulling it back in!)

- this is the usual $\vec{F}_{mag} = I\int d\vec{l} \times \vec{B}$ force on wires with current. ($F_{mag} = ILB$ here)
- this current is creating a "drag force", you have to actively pull the wire loop out otherwise it will come to rest.

- To maintain a steady speed, v, you <u>need</u> an external force, $\vec{F}_{ext} = ILB$ to the right!

Power by external force, $P_{ext} = \vec{F}_{ext} \cdot \vec{v}$

$$= +ILBV$$

$$P_{ext} = BLv\left(\frac{\mathcal{E}}{R}\right) = \frac{B^2L^2v^2}{R}$$

How much Energy is dissipated by the resistor (per unit time)?

$$P_{diss} = I^2R = \left(\frac{BLV}{R}\right)^2 R = \frac{B^2L^2V^2}{R}$$

The power input by the external pull is equal to that dissapated through the resistor.

★ Energy is conserved; it's the EMF that drives the current, but it's the pulling force that supplies the energy!

In general, these conducting loops resist changes in flux (Lenz's Law) by producing currents that oppose the change

Useful Applications: Magnetic Braking (trains, Prius),
Vending Machines (coin checker)
Inductive Heating (manufacturing / welding)

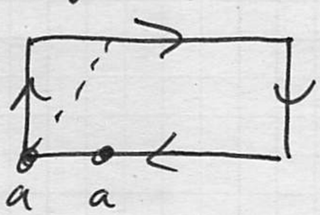
A curiosity: It appears that in this situation the magnetic field is doing work on charges!!

Griffiths points out that (7.1.3) the equation $\mathcal{E} = \oint \vec{f} \cdot d\vec{l}$ is taken at a specific time (a snapshot)

But, the work done/charge follows a charge ~~around~~ around the loop.

$$\frac{\text{Work done}}{\text{Charge}} = \oint \vec{f} \cdot d\vec{l}$$ following a charge around the loop.

the paths here can be quite different,

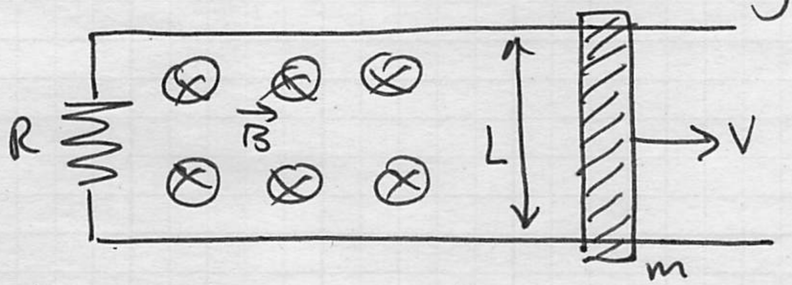


- the loop will have moved as we follow the charge around so the path it takes is a bit different.

Griffiths goes into detail about how the magnetic field does no work on the charges (7.1.3) and yet we can heat the resistor. The magnetic field & the physical wire both provide forces on the charges; it's the physical wire (and E-field) that do work w/ energy input by the external pull. Read 7.1.3 carefully!

Example: Sliding Bar

Consider a bar attached to two metal rails, but free to slide. The metal rails are L apart and are connected by a resistor, R.



A uniform magnetic field points into the page everywhere.

The bar is initially moving to the right and a current flows counterclockwise.

The current is,

$$I = \frac{\mathcal{E}}{R} = \frac{vBL}{R}$$ just as it has been.

There is not an active push by an external agent, so the bar will eventually stop moving. Why?

There's a magnetic drag force on it!

A current I runs through the bar giving a force to the left,

$$\vec{F}_{mag} = I\int d\vec{l} \times \vec{B} = BIL \text{ to the left}$$

(check with r.h. rule!)

$$F_{mag} = \frac{vB^2L^2}{R}$$ in terms of what we know.

Notice its a linear drag force!

If the bar is given an initial kick, so that it moves with v_0 at $t=0$ we can determine how it slows down, namely $v(t)$.

$$F_{net} = ma = m\frac{dv}{dt}$$ here, $F_{net} = -\frac{B^2L^2}{R}v$

minus b/c to the left!

$$m\frac{dv}{dt} = -\frac{B^2L^2}{R}v$$

$$\frac{dv}{v} = -\frac{B^2L^2}{mR}dt$$ integrate to $v(t)$ in a time t.

$$\int_{v_0}^{v(t)} \frac{dv}{v} = -\frac{B^2L^2}{mR}\int_0^t dt \Rightarrow \ln(v(t)) - \ln(v_0) = -\frac{B^2L^2}{mR}t$$

So that, $\ln\left(\frac{v(t)}{v_0}\right) = -\frac{B^2L^2}{mR}t$

thus,

$$v(t) = v_0 e^{-\frac{B^2L^2}{mR}t}$$

Let's check this, as $t \to \infty$ $v \to 0$ ✓

$\frac{B^2L^2}{mR}$ should have units of $\frac{1}{\text{time}}$, does it?

$$\left[\frac{B^2L^2}{mR}\right] = \frac{T^2m^2}{kg\,\Omega^2} = \frac{kg^2}{C^2s^2}\frac{m^2}{kg}\frac{sC^2}{kg\,m^2} = \frac{1}{s} \checkmark$$

the initial kinetic energy of the bar is $\frac{1}{2}mv_0^2$. where does the energy go?

To the resistor!

$$P = I^2R = \frac{\mathcal{E}^2}{R^2}R = \frac{\mathcal{E}^2}{R} = \frac{v^2B^2L^2}{R}$$ in any instant so $v = v(t)$

$$W = \int P\,dt = \frac{B^2L^2}{R}\int_0^\infty v^2(t)\,dt$$ have to wait the whole time $t \to \infty$.

$$W = \frac{B^2L^2}{R}\int_0^\infty V_0^2 e^{-2\frac{B^2L^2}{mR}t}\,dt$$

let $\alpha = 2\frac{B^2L^2}{mR}$ so,

$$W = \frac{B^2L^2}{R}V_0^2\int_0^\infty e^{-\alpha t}\,dt = \frac{B^2L^2}{R}\frac{V_0^2}{(-\alpha)}e^{-\alpha t}\Big|_0^\infty$$

$$= \frac{B^2L^2}{R}\frac{V_0^2}{\alpha}\left(e^0 - e^{-\infty}\right) = \frac{B^2L^2V_0^2}{R\alpha}$$

so, $$W = \frac{B^2L^2V_0^2}{R\alpha} = \frac{B^2L^2V_0^2}{R}\left(\frac{1}{2}\frac{mR}{B^2L^2}\right) = \frac{1}{2}mv_0^2 \checkmark$$