We can summarize the Lorentz Transformations with a powerful new notation.

Usually vectors in 3D ("3-vectors") have three components like $\vec{x}$ has $\begin{cases} x^1 = x \\ x^2 = y \\ x^3 = z \end{cases}$

(We write the component index as superscript for reasons that will readily become apparent.)

The idea is that we add a <u>fourth</u> <u>component</u>, ct, to the position vector.

$\Rightarrow$ You could call it component #4 (some do!) but most call it the "zero-th" component. so,

$$x^0 \equiv ct \qquad \text{with} \quad t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

so,

$$ct' = \gamma\left(ct - \frac{v}{c}x\right)$$

people often use $\beta \equiv v/c$ so $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

From this, we get,

$$\begin{aligned} ct' &\longleftrightarrow & x^{0'} &= \gamma(x^0 - \beta x^1) \\ x' &\longleftrightarrow & x^{1'} &= \gamma(x^1 - \beta x^0) \\ y' &\longleftrightarrow & x^{2'} &= x^2 \\ z' &\longleftrightarrow & x^{3'} &= x^3 \end{aligned}$$

we also use
$x' = \gamma(x - vt)$
$x' = \gamma(x - \frac{v}{c}ct)$

This is called a transformation, or "boost", because we're shifting to a frame "boosted" by velocity $v\hat{x}$.

We can summarize this using Matrix notation,

$$\begin{pmatrix} \bar{x}^{0'} \\ \bar{x}^{1'} \\ \bar{x}^{2'} \\ \bar{x}^{3'} \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{This matrix is often called "}\Lambda\text{"}} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

so more compactly, $(\bar{x}')^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}_{\ \nu} x^{\nu}$

Here, $(\text{Matrix})^{\mu \leftarrow \text{row element}}_{\nu \leftarrow \text{column element}}$

→ our superscript $\left[(\bar{x}')^{\mu} \text{ or } x^{\nu} \right]$ implies these are <u>column</u> <u>vectors</u>.

If we want <u>row</u> vectors, we'll lower the superscript to get subscripts (but that has implications!)

This notation should remind us of rotations in 3 space.

- Any <u>object</u> that transforms under spatial rotations like $\vec{r}$ does (e.g. $\vec{p}$) is a 3-vector.
- Any <u>object</u> that transforms under a Lorentz boost like x^{μ} does is called a "contravariant 4-vector" or simply "4-vector". (there are others).

Example: Displacement $\Delta X \equiv X_A - X_B$

* this is a contravariant 4-vector

Proof: $\Delta X^\mu = X_A^\mu - X_B^\mu$

If we boost the frame,

$$(\bar{X}_A')^\mu = \sum_{\nu=0}^{3} \Lambda^\mu_\nu X_A^\nu$$

(or Einstein notation $\Lambda^\mu_\nu X_A^\nu$ sum over repeated indicies)

$$(\bar{X}_B')^\mu = \Lambda^\mu_\nu X_B^\nu$$

So, $$(\Delta\bar{X}')^\mu = (\bar{X}_A')^\mu - (\bar{X}_B')^\mu = \Lambda^\mu_\nu (X_A^\nu - X_B^\nu)$$

$$(\Delta\bar{X}')^\mu = \Lambda^\mu_\nu \Delta X^\nu$$

So ΔX tranforms exactly as X did!

Why do we care?

→ we will see shortly. Not every collection of 4 things is a 4 vector!

Allude to future work!

Co-variant 4-vectors X_μ or (X_0, X_1, X_2, X_3)

"co" for "low" the index is a subscript, but these are row vectors. We haven't defined them yet.

The search for Invariant Quantities

- Invariant quantities – ones that are the same regardless of reference frame are helpful in a variety of ways. They can tell us about conservation principles, they can help us check our physics, and they can be starting points for sense making.

You're already encountered some, which work under Galilean Relativity,

$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$ is an invariant quantity in Galilean R.

so is

$s^2 = x^2 + y^2 + z^2$, which is arguably a special case of the above ($\vec{a} = \vec{r}$ and $\vec{b} = \vec{r}$).

Let's work by analogy, the dot product in 3 space is,

$$\vec{a} \cdot \vec{b} \equiv \sum_{\nu=1}^{3} a_\nu b^\nu$$ or simply $a_\nu b^\nu$ in Einstein notation

or $a_\mu b^\mu$ ← sum over any repeated index.

We <u>defined</u> the dot product in 3 space. Is the definition of any use when we have 4 vectors?

Consider $X^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$

In the boosted frame,

$$\bar{X}^\mu = \begin{pmatrix} c\bar{t} \\ \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \bar{x^0} \\ \bar{x^1} \\ \bar{x^2} \\ \bar{x^3} \end{pmatrix} = \begin{pmatrix} \gamma x^0 - \gamma\beta x^1 \\ \gamma x^1 - \gamma\beta x^0 \\ x^2 \\ x^3 \end{pmatrix}$$

This might appear odd but consider an operation where we take $\vec{X}_\mu X^\mu$, but do so comparing to the product $\bar{X}_\mu \bar{X}^\mu$ using our definition,

$a_\nu b^\nu$ as the dot product.

$$\bar{X}_\mu \bar{X}^\mu = \bar{X^0}^2 + \bar{X^1}^2 + \bar{X^2}^2 + \bar{X^3}^2$$

Now $\bar{X^2}^2 = (X^2)^2$ & $\bar{X^3}^2 = (X^3)^2$ so let's just focus on $\bar{X^0}^2 + \bar{X^1}^2$!

$$\rightarrow \bar{X^0}^2 + \bar{X^1}^2$$

$$= (\gamma x^0 - \gamma\beta x^1)^2 + (\gamma x^1 - \gamma\beta x^0)^2$$

$$= (\gamma^2 x^{0\,2} - 2\gamma\beta x^1 x^0 + \gamma^2\beta^2 x^{1\,2}) + (\gamma^2 x^{1\,2} - 2\gamma\beta x^1 x^0 + \gamma^2\beta^2 x^{0\,2})$$

What a mess!

But if we <u>defined</u> the first term to be negative then a lot of terms would cancel out,

$$-\bar{X^0}^2 + \bar{X^1}^2 = \gamma^2 x^{0\,2}(-1+\beta^2) + \gamma^2 x^{1\,2}(-\beta^2+1)$$

All the $x^0 x^1$ terms cancel!

Ok that's weird why would we do that.

We get to <u>define</u> the scalar product, so let's where it takes us.

$$\gamma^2 = \frac{1}{1-\beta^2} \quad \text{as we define} \quad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

so $\gamma^2(1-\beta^2) = 1$ so that,

$$-\bar{x}^{0\,2} + \bar{x}^{1\,2} = \gamma^2(1-\beta^2)\left[-x^{0\,2}\right] + \gamma^2(1-\beta^2)\left[x^{1\,2}\right]$$

$$= -x^{0\,2} + x^{1\,2}$$

Whoa! That's really nice b/c with this definition, this quantity appears invariant!

ok so what are we doing formally?

$x_0 = -x^0 = -ct$ — the covariant form of the zeroth component has a negative sign (only the zeroth.)

If we do this then,

$a_\mu b^\mu$ gets that minus sign only in the $a_0 b^0$ term (in our case the $x_0 x^0$ term)

thus,

$$x \cdot x \equiv x_\mu x^\mu = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$$

$$\bar{x} \cdot \bar{x} \equiv \bar{x}_\mu \bar{x}^\mu = -(\bar{x}^0)^2 + (\bar{x}^1)^2 + (\bar{x}^2)^2 + (\bar{x}^3)^2$$

But $-\bar{x}^{0\,2} + \bar{x}^{1\,2} = -x^{0\,2} + x^{1\,2}$ and $x^2 = \bar{x}^2$, $x^3 = \bar{x}^3$

So $x \cdot x$ is <u>invariant</u> you will get the same answer in every frame. (It's easy to show $a_\mu b^\mu = \bar{a}_\mu \bar{b}^\mu$ in general)

Lorentz 4 vectors Transform according to the following operation,

$$\bar{a}^{\mu} = \Lambda^{\mu}_{\ \nu}\, a^{\nu}$$

We have found that $X^{\nu} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$ and ΔX^{ν} are both 4 vectors that transform this way

Claim: multiply & dividing X^{ν} or ΔX^{ν} (or any 4 vector) by a Lorentz invariant quantity gives you a 4 vector.

Let's see how, consider a "4-velocity"

$$u^{\nu} = \frac{\Delta X^{\nu}}{\Delta \tau}$$

where $\Delta\tau$ is the proper time a quantity that is Lorentz invariant (i.e. all observers will agree on its value)

Let's see how u^{ν} transforms,

$$\bar{u}^{\mu} \overset{?}{=} \Lambda^{\mu}_{\ \nu}\, u^{\nu}$$

We will focus on just ΔX^0 & ΔX^1 as ΔX^2 & ΔX^3 are the same.

$$\Lambda^{\mu}_{\ \nu}\, u^{\nu} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} \Delta X^0/\Delta\tau \\ \Delta X^1/\Delta\tau \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} c\Delta t/\Delta\tau \\ \Delta x/\Delta\tau \end{pmatrix}$$

$$\Lambda^\mu_{\ \nu} u^\nu = \begin{pmatrix} \gamma \frac{c\Delta t}{\Delta\tau} - \gamma\beta \frac{\Delta x}{\Delta\tau} \\ -\gamma\beta \frac{c\Delta t}{\Delta\tau} + \gamma \frac{\Delta x}{\Delta\tau} \end{pmatrix}$$

Now all observers agree on $\Delta\tau$, so it will not be altered by this transformation.

So,

$$\Lambda^\mu_{\ \nu} u^\nu = \frac{1}{\Delta\tau} \begin{pmatrix} \gamma c\Delta t - \gamma\beta\Delta x \\ -\gamma\beta c\Delta t + \gamma\Delta x \end{pmatrix} = \frac{\Delta \bar{x}^\mu}{\Delta\tau} = \bar{u}^\mu$$

thus,

$\bar{u}^\mu = \Lambda^\mu_{\ \nu} u^\nu$ is just $\frac{1}{\Delta\tau}\Delta\bar{x}^\mu = \frac{1}{\Delta\tau}\Lambda^\mu_{\ \nu}\Delta x^\nu$

that is, we've just divided both sides by the Lorentz invariant $\Delta\tau$.

Let's say we instead used Δt the time between the events in the S frame,

$$\Lambda^\mu_{\ \nu} \frac{\Delta x^\nu}{\Delta t} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} c \\ \Delta x/\Delta t \end{pmatrix}$$

$$= \begin{pmatrix} \gamma c - \gamma\beta \frac{\Delta x}{\Delta t} \\ -\gamma\beta c + \gamma \frac{\Delta x}{\Delta t} \end{pmatrix}$$

here it is fair to call $\frac{\Delta x}{\Delta t} = u_x$ the speed in the S frame.

Is the second term the speed in the $\bar{S}$ frame? We found $\bar{u}_x = \frac{u_x - v}{1 - u_x v/c^2}$

Let's assume we obtained,

$\begin{pmatrix} \Delta\bar{x}^0/\Delta\bar{t} \\ \Delta\bar{x}^1/\Delta\bar{t} \end{pmatrix}$ so that $\bar{u}_x = \frac{\Delta\bar{x}^1}{\Delta\bar{t}}$

If this transformation worked we should get

$$\bar{u}_x = \frac{u_x - v}{1 - u_x v/c^2} \qquad \text{from} \qquad \bar{u}_x = -\gamma\beta c + \gamma \frac{\Delta x}{\Delta t}$$

$$\bar{u}_x = -\gamma\beta c + \gamma u_x = -\frac{v}{c} c \Big/ \sqrt{1 - v^2/c^2} + \frac{v}{c} u_x$$

$$= \frac{u_x v}{c} - \frac{v}{\sqrt{1 - v^2/c^2}}$$

$$\bar{u}_x = \frac{\sqrt{1 - v^2/c^2}\, u_x v - vc}{c\sqrt{1 - v^2/c^2}} = \frac{\left(\sqrt{1 - v^2/c^2}\, u_x - c\right) v}{c\sqrt{1 - v^2/c^2}}$$

hmmm.... seems real hard to get this ↑ to be this → $\bar{u}_x = \frac{u_x - v}{1 - u_x v/c^2}$ and I'd expect to be able to tranform a velocity component this way.... what we're done with $\Delta x/\Delta t$ must not transform the way a 4-vector does. Thus, $\Delta x/\Delta t$ is not a 4 vector in the same way we mean.

Summary: Covariant 4-vector $x_\mu \equiv (x_0, x_1, x_2, x_3)$
$= (-x^0, x^1, x^2, x^3)$
$= (-ct, x, y, z)$

Key point: $x_0 = -x^0$ so that,

$$\sum_\mu x_\mu x^\mu = -c^2t^2 + x^2 + y^2 + z^2$$

→ defined so that $a_\mu b^\mu$ is Lorentz invariant.

Example #1: Turn on a point source of light at $x=y=z=t=0$ in frame S.

At a time t in S, I'd expect a ~~[illegible]~~ spherical wave front, $x^2+y^2+z^2 = c^2t^2$

What about in the frame $\bar{S}$ that moves at speed v?

$$\left.\begin{aligned} \bar{x} &= \gamma(x - vt) \\ \bar{y} &= y \\ \bar{z} &= z \\ \bar{t} &= \gamma(t - vx/c^2) \end{aligned}\right\}$$

This will give:

$$\bar{x}^2 + \bar{y}^2 + \bar{z}^2 = c^2\bar{t}^2$$

a spherical wavefront in $\bar{S}$.
(Speed of light is c everywhere in vacuum in every frame.)

So the formula

$x_\mu x^\mu = 0$ describes a wave front in any frame.

The formula is written in a manifestly invariant way. (it's frame independent). Length and time are not frame independent, but the relationship that describes the wave front is.

Example #2: Consider Δx^μ (which is a 4 vector)

$$\Delta x_\mu \Delta x^\mu = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$
$$= \underbrace{d^2}_{\substack{\text{spatial separation}\\ \text{between two events}}} - c^2 \underbrace{\Delta t^2}_{\substack{\text{time separation between}\\ \text{those same two events.}}}$$

Neither d nor Δt is observer independent, we ~~[illegible]~~ showed that early on with our work in time dilation and length contraction.

But it turns out that the combination is Lorentz Invariant!

We define this to be the "space time interval,"

$$I \equiv \Delta X_\mu \Delta X^\mu = (a_\mu - b_\mu)(a^\mu - b^\mu)$$
$$= a_\mu a^\mu - a_\mu b^\mu - b_\mu a^\mu + b_\mu b^\mu$$

Notice each term has the form $a_\mu b^\mu$ so we know I must be Lorentz invariant as each term is.

Again this manifestly invariant. (like $x_\mu x^\mu$)

It is important in it's own right so let's unpack it a bit more.

Scenario 1: Two events occur at one location in some frame.

(turn a light on, wait, turn it off)

event 1 $\longrightarrow$ event 2

$$I_{\substack{\text{these 2}\\ \text{events}}} = d^2 - c^2\Delta t^2 = 0 - c^2\Delta t^2 < 0$$

While we derived $I<0$ in this frame, every observer in any frame.

These events are called "time like separated events" or "time like" for short. Because they are (in some frame only) separated in time.

Because $I<0$ in every frame, no observer in any frame could claim these events are simultaneous; one always occurs first.

For time like separated events, we can always find a frame moving with $v<c$ where the events occur at the same location. (also any time $d^2 < c^2\Delta t^2$)

Scenario 2: 2 events occur simultaneously but at different locations in some frame

(turn on two lights at different ends of a room).

$$I_{\substack{\text{these 2}\\ \text{events}}} = d^2 - c\Delta t^2 = d^2 - 0 > 0$$

Again any observer in any frame will find $I>0$.

These are "space like separated events" or "space like" for short, because in some frame they are separated in space only.

Because $I>0$, no observer in any frame will claim that these occur in the same place, they are always separated in space.

For space like events, we can always find a frame moving with $v<c$ where the events occur at the <u>same time</u>. (also any time $d^2 > c^2\Delta t^2$)

<u>Scenario 3:</u> If $I=0$, $d^2=c^2\Delta t^2$.

This called "light like" separation because a beam of light could go from one event to the other (it could travel d in a time Δt).

Note: If two events A+B are time like separated, $I_{AB}<0$ there's a frame where $d=0$.

Suppose in frame S they are not at the same place, $\Delta x \neq 0$.

In $\bar{S}$, $\Delta\bar{x} = \gamma(\Delta x - v\Delta t)$.

If we choose $V = \Delta x/\Delta t$ then $\Delta\bar{x}=0$ (Defines the frame)

This is ok because, $I_{AB} = \Delta x^2 - c^2\Delta t^2 < 0$

$\Delta x < c\Delta t \Rightarrow V = \frac{\Delta x}{\Delta t} < c$

which is fine.

But with a space like event, there's no frame where $\Delta x=0$.

$\Delta x^2 - c^2\Delta t^2 > 0$ so $\Delta x > c\Delta t$ thus $V = \frac{\Delta x}{\Delta t} > c$

<u>not possible.</u>

we get complex γ .... ugh.