

## Wave Equation for vector quantities

- We've seen the 1D wave equation:  $\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$   
 that gives rise to solutions:  $f(x-vt)$

We have sinusoidal solutions of the form:

$$\tilde{f} = \tilde{A} e^{i(kx - \omega t)}$$

- We've seen the 3D wave equation:  $\nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$   
 that gives rise to solutions:  $f(\vec{r} - \vec{v}t)$

We have sinusoidal solutions of the form:

$$\tilde{f} = \tilde{A} e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

What about if the disturbance is a vector itself?  
 (e.g.  $\vec{E}$  or  $\vec{B}$ ), then "f" is a vector quantity and  
 our solutions will look like,

$$\vec{f}(\vec{r}, t) = f_x(\vec{r}, t) \hat{x} + f_y(\vec{r}, t) \hat{y} + f_z(\vec{r}, t) \hat{z}$$

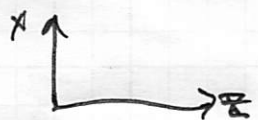
each of these is a wave

Our notation becomes a little nasty  $\vec{f}_{\text{physical}}(\vec{r}, t) = \text{Re}[\vec{f}(\vec{r}, t)]$

Note: if  $\vec{f}$  is  $\perp$  to  $\vec{k}$ , the wave is "transversely polarized"  
 or "transverse"

if  $\vec{f}$  is  $\parallel$  to  $\vec{k}$ , the wave is "longitudinally polarized".

For example, a wave moving in  $z$ -direction but vertically polarized (in  $x$ ) is transverse  
 or "longitudinal"



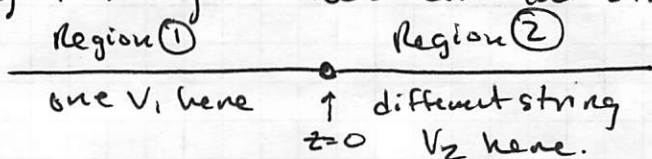
$$\vec{f}(z, t) = \tilde{A} e^{i(kz - \omega t)} \hat{x}$$

If  $\vec{f}$  is transverse and  $\vec{f}(\vec{r}, t)$  always points in the same  
 direction (as above), this wave is linearly polarized

## Prelude to Waves & Boundaries

At a boundary between two media (strings, materials, etc.), a wave will typically be transmitted (into the new media) and reflected (into the old media).

In 1D you might imagine two different strings fused at a point.



An incident wave may come in from the left,

$$\tilde{f}_I(z,t) = \tilde{A}_I e^{i(k_1 z - \omega t)} \quad \text{with} \quad \begin{cases} z < 0 \text{ only} \\ k_1 > 0 \text{ (moves right)} \\ v_1 = \omega/k_1 \end{cases}$$

But, this will be superposed with

the reflected wave, (left moving!)

$$\tilde{f}_R(z,t) = \tilde{A}_R e^{i(-k_1 z - \omega t)} \quad \text{with} \quad \begin{cases} z < 0 \text{ only} \\ k_1 > 0 \text{ still} \\ |v_1| = \omega/k_1 \text{ moves left.} \end{cases}$$

In region 2, there will be a transmitted wave,

$$\tilde{f}_T(z,t) = \tilde{A}_T e^{i(k_2 z - \omega t)} \quad \text{with} \quad \begin{cases} z > 0 \\ k_2 > 0 \\ v_2 = \omega/k_2, \text{ right moving} \end{cases}$$

Claim!  $\omega$  is the same for all these waves!

$\omega$  counts the wiggles/second and the boundary point has one wiggles frequency,  $\omega$ .

Boundary Conditions will require that  $f(z,t) + \frac{df}{dz}(z,t)$  be continuous. Soon we will find  $A_T$  and  $A_R$  given  $A_I$

## Electromagnetic Waves

Without Sources ( $\rho=0$ ;  $\vec{J}=0$ ), Maxwell's equations are, (in vacuum)

$$\begin{aligned}\nabla \cdot \vec{E} &= 0 & \nabla \times \vec{E} &= -\frac{d\vec{B}}{dt} \\ \nabla \cdot \vec{B} &= 0 & \nabla \times \vec{B} &= \mu_0 \epsilon_0 \frac{d\vec{E}}{dt}\end{aligned}$$

We will take the curl of Faraday's Law, we will find a 3D wave equation for  $\vec{E}$ ,

$$\nabla \times (\nabla \times \vec{E}) = -\frac{d}{dt} (\nabla \times \vec{B})$$

$$\underbrace{\nabla(\nabla \cdot \vec{E})}_{0} - \nabla^2 \vec{E} = -\frac{d}{dt} (\mu_0 \epsilon_0 \frac{d\vec{E}}{dt})$$

$$-\nabla^2 \vec{E} = -\mu_0 \epsilon_0 \frac{d^2 \vec{E}}{dt^2} \Rightarrow \boxed{\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{d^2 \vec{E}}{dt^2}}$$

If we take  $\nabla \times (\nabla \times \vec{B})$ , we will get,

$$\boxed{\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{d^2 \vec{B}}{dt^2}}$$

That is both  $\vec{E}$  &  $\vec{B}$  satisfy 3D wave equations in vacuum.

In Cartesian Coordinates for  $\vec{E}$ ,

$$\nabla^2 \vec{E} \text{ means } \frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2}$$

So what we really have are six! equations (3 for  $\vec{E}$  & 3 for  $\vec{B}$ )

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2} \equiv \frac{1}{v^2} \frac{\partial^2 E_x}{\partial t^2}$$

We have 5 more of these for

$$E_y, E_z, B_x, B_y, \& B_z!$$



Each component of  $\vec{E}$  &  $\vec{B}$  satisfy a 3D wave equation with speed  $v = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 2.998 \cdot 10^8 \text{ m/s}$

- We've seen this before, we can have wave-like solutions as long as they travel at a speed  $= c$  in vacuum!
- This suggests that light could be a wave but it's not yet proven here.

The solutions to the wave equations are very general and depend very much on initial conditions and boundary conditions.

We will limit our scope to the simplest, idealized solutions; plane wave solutions.

→ We can build up other solutions by superposing these solutions using the method of Fourier.

So we will investigate monochromatic plane wave solutions, (fixed  $\omega$ )

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$\vec{E}_0$ : amplitude vector  
 $\vec{k}$ : wave vector; wave travel in  $\vec{k}$  direction  
 $\omega$ : frequency;  $\omega = c|k|$

Complex  $\vec{E}$

the real physical

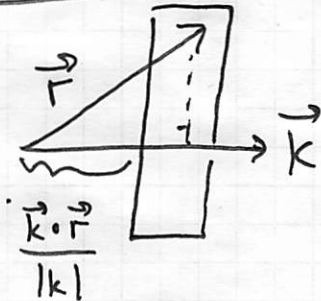
$\vec{E}$  is given by  $\text{Re}[\vec{E}]$ .

• It's constant!

• But, it has a phase  $e^{i\phi}$

How do we make plane waves?

Claim! If  $\vec{k} \cdot \vec{r} = \text{constant}$ , that describes a plane



Any point on this plane will

have the same value of  $\vec{k} \cdot \vec{r}$

So that means  $E$  is the same value for all points on this plane ( $\perp$  to  $\vec{k}$ )

Let's assume  $\vec{k} \cdot \vec{r} = 0$ , then we have a very

simple plane wave,  $\vec{E} = \vec{E}_0 e^{i\omega t}$

fixed const.  $\leftarrow$

$$\vec{E}_0 = \tilde{E}_{0x} \hat{x} + \tilde{E}_{0y} \hat{y} + \tilde{E}_{0z} \hat{z}$$

We've shown that Maxwell's Equations have wave solutions in free space, but if you pick some solution, you must check that the solution is consistent w/ all the Maxwell equations.

$\Rightarrow$  Doing this check will lead us to certain conditions on  $\vec{E}$  &  $\vec{B}$  (and  $\vec{k}$ !)

We will start this work by assuming a general plane wave solution.

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

and then impose Maxwell's Equations on it.

## Conditions on Plane Wave Solutions

start with  $\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$

Impose Gauss' Law  $\nabla \cdot \vec{E} = 0$ ,

$$\begin{aligned} \nabla \cdot \vec{E} &= \nabla \cdot \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} = \frac{\partial}{\partial x} \tilde{E}_{0x} e^{i(k_x x + k_y y + k_z z - \omega t)} \\ &\quad + \frac{\partial}{\partial y} \tilde{E}_{0y} e^{i(k_x x + k_y y + k_z z - \omega t)} \\ &\quad + \frac{\partial}{\partial z} \tilde{E}_{0z} e^{i(k_x x + k_y y + k_z z - \omega t)} \end{aligned}$$

$$\begin{aligned} \nabla \cdot \vec{E} &= i k_x \tilde{E}_{0x} e^{i(\vec{k} \cdot \vec{r} - \omega t)} \\ &\quad + i k_y \tilde{E}_{0y} e^{i(\vec{k} \cdot \vec{r} - \omega t)} + i k_z \tilde{E}_{0z} e^{i(\vec{k} \cdot \vec{r} - \omega t)} \end{aligned}$$

$$\nabla \cdot \vec{E} = i(\vec{k} \cdot \vec{E}) e^{i(\vec{k} \cdot \vec{r} - \omega t)} = 0$$

if this is always true then  $\vec{k} \cdot \vec{E} = 0!$

This means that  $\vec{E}$  must be perpendicular to  $\vec{k}$ , that is, the electric field is transversely polarized (in vacuum).

(Note: Fourier summing cannot change this!)

if we assume that  $\vec{B}$  has the same complex form,

$$\nabla \cdot \vec{B} = 0 \text{ tells us that } \vec{k} \cdot \vec{B} = 0 \text{ also so } \vec{B} \text{ must also be transverse!}$$

Let's impose Faraday's Law,  $\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$  should connect  $\vec{E} \perp \vec{B}$ .

$$\begin{aligned} \nabla \times \vec{E} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \tilde{E}_x & \tilde{E}_y & \tilde{E}_z \end{vmatrix} = \hat{x} \left( \frac{\partial}{\partial y} \tilde{E}_z - \frac{\partial}{\partial z} \tilde{E}_y \right) \\ &\quad - \hat{y} \left( \frac{\partial}{\partial x} \tilde{E}_z - \frac{\partial}{\partial z} \tilde{E}_x \right) \\ &\quad + \hat{z} \left( \frac{\partial}{\partial x} \tilde{E}_y - \frac{\partial}{\partial y} \tilde{E}_x \right) \end{aligned}$$



Given that  $\vec{E}_x = \vec{E}_{0x} e^{i(\vec{k}\cdot\vec{r}-\omega t)}$  and similar for  $\vec{E}_y + \vec{E}_z$ ,

$$\begin{aligned}\nabla \times \vec{E} &= \hat{x} (ik_y \vec{E}_{0z} - ik_z \vec{E}_{0y}) e^{i(\vec{k}\cdot\vec{r}-\omega t)} \\ &+ \hat{y} (ik_x \vec{E}_{0z} - ik_z \vec{E}_{0x}) e^{i(\vec{k}\cdot\vec{r}-\omega t)} \\ &+ \hat{z} (ik_x \vec{E}_{0y} - ik_y \vec{E}_{0x}) e^{i(\vec{k}\cdot\vec{r}-\omega t)} \\ &= i(\vec{k} \times \vec{E}_0) e^{i(\vec{k}\cdot\vec{r}-\omega t)}\end{aligned}$$

(the arguments in parentheses are components of this cross product!)

$$-\frac{d\vec{B}}{dt} = -\frac{d}{dt} \vec{B}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = i\omega \vec{B}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

So Faraday's Law suggests,  $\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$ ,

$$i(\vec{k} \times \vec{E}_0) e^{i(\vec{k}\cdot\vec{r}-\omega t)} = i\omega \vec{B}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

Thus we find that  $\vec{k} \times \vec{E}_0 = \omega \vec{B}_0$ , so

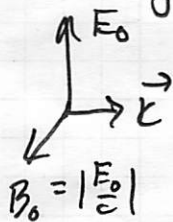
$\vec{E} + \vec{B}$  are transverse and perpendicular to each other!

For example, if  $\vec{k} \Rightarrow \hat{z}$  direction,

$$\vec{B}_0 = \frac{k}{\omega} (\hat{z} \times \vec{E}_0) = \frac{1}{c} (\hat{z} \times \vec{E}_0)$$

→ Both  $\vec{E} + \vec{B}$  are transverse;  $\vec{B} + \vec{E}$  are perpendicular to each other and  $|B_0| = \frac{1}{c} |E_0|$

→ Finally  $\vec{B}_0$  is in phase with  $\vec{E}_0$  (no complex issues!)



General traveling wave description

Note: we relied on complex analysis to simplify things.

$$\nabla \times \vec{A}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = i\vec{k} \times \vec{A}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\nabla \cdot \vec{A}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = i\vec{k} \cdot \vec{A}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\frac{d}{dt} \vec{A}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = -i\omega \vec{A}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

There's one last Maxwell Equation, but it doesn't give any more constraints (it's redundant)

So our full plane wave solutions that obey all 4 Maxwell Equations in vacuum are:

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \hat{n} \quad \hat{n} \text{ is the polarization direction of } \vec{E}_0, \hat{n} \cdot \vec{k} = 0$$

$$\vec{B}(\vec{r}, t) = \vec{B}_0 e^{+i(\vec{k} \cdot \vec{r} - \omega t)} (\hat{k} \times \hat{n}) = \frac{\vec{E}_0}{c} e^{i(\vec{k} \cdot \vec{r} - \omega t)} (\hat{k} \times \hat{n}) = \frac{\hat{k} \times \vec{E}_0}{c}$$

Remember that the physical  $\vec{E}$  is,

$$\vec{E}_{\text{physical}}(\vec{r}, t) = \text{Re}[\vec{E}] = \vec{E}_0 \cos(\vec{k} \cdot \vec{r} - \omega t + \delta)$$

(the  $\delta$  is hidden in the  $\vec{E}_0$ )

$$\vec{B}_{\text{physical}} = \frac{\vec{k} \times \vec{E}_{\text{physical}}}{c}$$