$\vec{S} = \frac{\text{energy flow}}{\text{per unit time (+ area)}}$ transported by $\vec{E}$ & $\vec{B}$ $= \frac{\vec{E}\times\vec{B}}{\mu_0}$

A local statement of energy conservation looks at densities,

Locally: $\frac{du_q}{dt} = \vec{E}\cdot\vec{J} = -\frac{\partial}{\partial t}u_{em} - \nabla\cdot\vec{S}$

this is poynting's theorem (derived in 1884)

We can reorganize this statement,

$$\frac{\partial}{\partial t}\left(u_q + u_{em}\right) = -\nabla\cdot\vec{S}$$

- u_q: this is Griffith's u_{mech}, particle's energy density (could be complicated KE obviously & thermal and other forms of PE)
- u_{em}: this is the energy density of the $\vec{E}$ & $\vec{B}$ fields
- $-\nabla\cdot\vec{S}$: this is the $\frac{\text{outflow}}{\text{volume}}$ of energy current. $\vec{S} = \vec{E}\times\vec{B}/\mu_0$

the statement,

$$\frac{\partial}{\partial t}\left(u_q + u_{em}\right) = -\nabla\cdot\vec{S}$$

is our classic conservation law structure

$$\frac{\partial}{\partial t}(\text{something}) = -\nabla\cdot\left(\text{that something's associated current density}\right)$$

$\vec{S}$ energy current density $= \frac{\text{flow of energy}}{\sec\cdot m^2}$

Compare this to,

$$\frac{\partial}{\partial t}(\rho) = -\nabla\cdot\vec{J} \qquad \vec{J} = \frac{\text{flow of charge}}{\sec\cdot m^2}$$

Globally: (integrating over a volume) we get back to

$$\frac{d}{dt}\iiint (u_q + u_{em})\, d\tau = -\iiint \nabla\cdot\vec{S}\, d\tau = -\oiint \vec{S}\cdot d\vec{A}$$

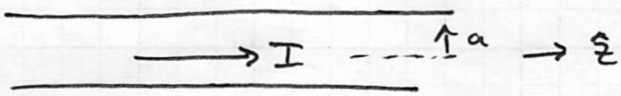
rate of increase of **all** energy = − (outflow of energy / second)

Side note:

In materials $\vec{S} = \vec{E}\times\vec{H}$ and

$$u_{em} = \frac{1}{2}\vec{E}\cdot\vec{D} + \frac{1}{2}\vec{H}\cdot\vec{B}$$

Example: Steady current in a wire



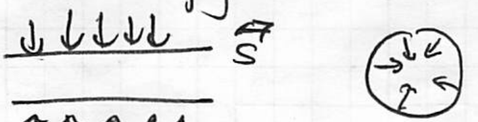
Consider a long wire with a steady current.

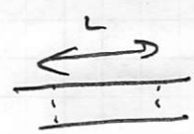
We know that $\vec{E} = E_0 \hat{z}$ and $\vec{J} = \sigma \vec{E} = \sigma E_0 \hat{z}$

As we have done in the past $\vec{B}_{inside} = \frac{\mu_0 J \pi r^2}{2\pi r}\hat{\varphi} = \frac{\mu_0 \sigma E_0}{2} r \hat{\varphi}$

At the edge $\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0} = \frac{\sigma E_0^2}{2} a (\underbrace{\hat{z}\times\hat{\varphi}}_{-\hat{s}})$

So the energy flows inwards!





Consider some length of wire, L

Across this length, $\Delta V = E_0 L$ and $I = J\pi a^2 = \sigma E_0 \pi a^2$

So, with $\frac{d}{dt}(W + U_{em}) = -\oiint \vec{S}\cdot d\vec{A}$

U_{em} is steady b/c neither $\vec{E}$ nor $\vec{B}$ change with time,

$\frac{dU_{em}}{dt} = 0$ so,

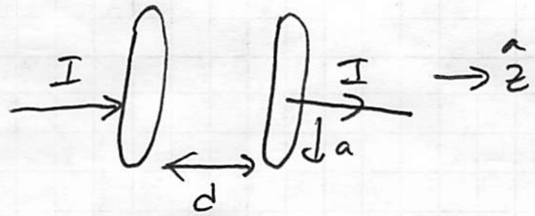
$$\frac{dW}{dt} = -\oiint \vec{S}\cdot d\vec{A} = -\frac{\sigma E_0^2}{2} a(\hat{s})\cdot(\underbrace{2\pi a L}_{\text{outer area}}\hat{s})$$

(note: end caps don't contribute)

$$= +\underbrace{(\sigma E_0 \pi a^2)}_{\text{current, } I}\underbrace{(E_0 L)}_{\text{potential diff, } \Delta V}$$

The total power entering the wire is $P = I\Delta V$! as we've always said. It enters via the fields! It's converted to $W(U_{mech}) \rightarrow$ thermal energy.

Example: A slowly (quasi-static) charging capacitor



We are going to investigate the energy as the capacitor charges up.

with $d \ll a$,
By Gauss' Law $\vec{E} = \frac{Q}{A\varepsilon_0}\hat{z}$ (and zero outside, right?)

By the Maxwell-Ampere Law, the magnetic field due to the wire is,

$$\oint \vec{B}(s)\cdot d\vec{l} = \mu_0 I \Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\varphi}$$

At the edge of the capacitor we showed that

$$\oint \vec{B}(s)\cdot d\vec{l} = \mu_0 \varepsilon_0 \iint \vec{J}_D\cdot d\vec{A}$$ gave us

$\vec{B}(a) = \frac{\mu_0 I}{2\pi a}\hat{\varphi}$ so the fields match there! remember?

At the edge of the capacitor ($s=a$),

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}\frac{Q}{A\varepsilon_0}\frac{\mu_0 I}{2\pi a}\underbrace{[\hat{z}\times\hat{\varphi}]}_{-\hat{s}}$$

so,

$$\vec{S} = \frac{-Q}{A\varepsilon_0}\frac{I}{2\pi a}\hat{s}$$

energy flows in as we charge!

The total energy out / time is,

$$\oiint \vec{S}\cdot d\vec{A}$$

this integral is taken in cylindrical coordinates just outside the capacitor.

$d\vec{A} = s\,d\phi\,dz\,\hat{s}$ (area points outward)

at surface of capacitor edge $s=a$

$$\oint\!\!\oint \vec{S}\cdot d\vec{A} = \frac{QI}{2\pi\varepsilon_0 aA}(-\hat{s})\cdot(\underbrace{2\pi a d}_{\text{just the outer area}})\hat{s}$$

$$= -\frac{QI}{\varepsilon_0}\frac{d}{A}$$

So the energy flows <u>into</u> the capacitor from external fields.

The stored energy between the plates is

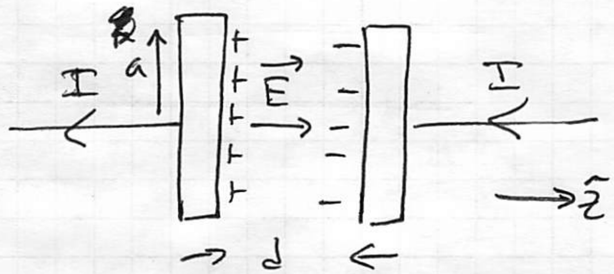
$$U_{em} = \left(\tfrac{1}{2}\varepsilon_0 E^2\right)\text{Volume} = \tfrac{1}{2}\varepsilon_0\left(\frac{Q}{A\varepsilon_0}\right)^2 (Ad)$$

(E ↑ constant field.)

So
$$\frac{dU_{em}}{dt} = \frac{2Q}{2\varepsilon_0}\frac{dQ}{dt}\frac{d}{A} = \frac{QI}{\varepsilon_0}\,d/A \quad \text{which is } \oint\!\!\oint \vec{S}\cdot d\vec{A}\,!$$

$$\text{increase of stored energy / sec} = \frac{\text{flow of energy in}}{\text{sec.}}$$

Example! A discharging Capacitor



A capacitor is connected to very long leads. It has a circular cross section, radius, a, and a separation, d. with $d \ll a$.

We intend to find $\vec{S}$ to see how the energy is transported.

Between the plates $\vec{E} = \frac{Q}{\pi a^2 \varepsilon_0}\hat{z}$ like usual for a capacitor.

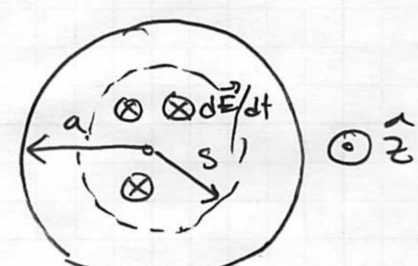
But now, $\frac{d\vec{E}}{dt}$ points in $-\hat{z}$! see why?

This also makes sense from a conservation of charge situation, $\frac{dQ}{dt} = -I$

Ok so we can compute $\vec{J}_D$,

$$\vec{J}_D = \varepsilon_0 \frac{d\vec{E}}{dt} = \frac{dQ/dt}{\pi a^2}\hat{z} = -\frac{I}{\pi a^2}\hat{z}$$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \iint \vec{J}_D \cdot d\vec{A}$$

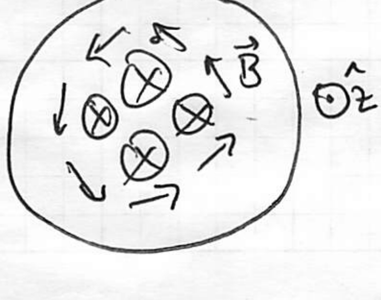


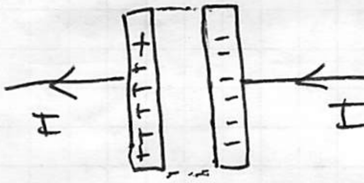
$$B 2\pi s = -\frac{\mu_0 I}{\pi a^2}\pi s^2$$

so,

$$\vec{B} = -\frac{\mu_0 I s}{2\pi a^2}\hat{\phi}$$

circulates opposite our other example. Makes sense b/c $d\vec{E}/dt$ points the other way.





$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

We will evaluate this at the surface of the dashed cylinder to see what the energy density current is doing.

$$\vec{S} = \frac{1}{\mu_0}\left(\frac{Q}{\pi a^2 \varepsilon_0} \hat{z} \times - \frac{\mu_0 I s}{2\pi a^2} \hat{\phi} \right)\Bigg|_{s=a}$$

$$= \frac{QIa}{2(\pi a^2)^2 \varepsilon_0} \left(\underbrace{\hat{z} \times -\hat{\phi}}_{+\hat{s}} \right) = \frac{QIa}{2(\pi a^2)^2 \varepsilon_0} \hat{s}$$

energy flows out of the region!

this is not Quasistatic! If RC circuit,

then $I(t) = \frac{V_0}{R} e^{-t/RC}$ and

$$Q(t) = CV_0 e^{-t/RC} \qquad \text{note: } C = \frac{A\varepsilon_0}{d}$$

$$\vec{S} = \frac{\left(\frac{V_0}{R}\right)\left(e^{-t/RC}\right)\left(\frac{A\varepsilon_0}{d} V_0\right)\left(e^{-t/RC}\right) a}{2A^2 \varepsilon_0} \hat{s}$$

$$\vec{S} = \frac{V_0^2}{R} \frac{a}{2Ad} e^{-2t/RC} \hat{s} \qquad \tau = \frac{RC}{2}$$

energy dissapation has time constant that is 1/2 that of I or Q.

B/c not Quasistatic

$$U_{em}(t) = \iiint \frac{\varepsilon_0}{2} E^2 d\tau + \iiint \frac{1}{2\mu_0} B^2 d\tau$$

is needed to find dU_{em}/dt!