We so far found 2 different ways of finding the magnetic field,

(1) Biot-Savart: $\vec{B} = \frac{\mu_0}{4\pi}\int \frac{I\,d\vec{l}\times\hat{\imath}}{\imath^2}$

which is analogous to Colomb's Law for $\vec{E}$

(2) Ampere's Law: $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$

which is analogous to Gauss's Law for $\vec{E}$

However, there was a third way of finding $\vec{E}$ which was suggested by the fact that

$\nabla\times\vec{E}=0 \longrightarrow \vec{E} = -\nabla V$ where $V = \frac{1}{4\pi\epsilon_0}\int\frac{\rho\,d\tau'}{\imath}$

the electric potential was a useful tool for determining the electric field, which often simplified and certainly diversified (e.g., separation of variables) the solution paths.

Can we find something similar for $\vec{B}$? YES!

The Vector Potential, $\vec{A}$

$\nabla\cdot\vec{B}=0$ suggests that $\vec{B}$ is the curl of some vector function, $\vec{A}$. B/c $\nabla\cdot(\nabla\times\vec{A})=0$ for any $\vec{A}$.

This function $\vec{A}$ is the "vector potential"

Why did we like potential functions?

V had some nice properties:

- it was a scalar (made things simpler)
- it satisfied Laplace's equation $\nabla^2 V = \rho/\epsilon_0$, but often $\nabla^2 V = 0$.

$\Rightarrow$ V would thus be "easy" to determine

$V = \frac{1}{4\pi\epsilon_0}\int\frac{\rho\,d\tau'}{\imath}$ or solve $\nabla^2 V = 0$

V did have some intrinsic ambiguity (could always add a constant)

But once you found V, $\vec{E}$ was simple to compute,
$\vec{E} = -\nabla V$ plus V had an energy/charge interpretation

$\Rightarrow$ We'd like to find something similar for $\vec{B}$

$\vec{A} \equiv$ vector potential

$\vec{A}$ isn't as nice on almost all counts,
- vector not a scalar
- not such a simple physical meaning

But $\vec{A}$ is still very useful in a number of contexts, so we will add it to our toolbox.

<u>Relating $\vec{A}$ to current density, $\vec{J}$</u>

$\nabla \times \vec{B} = \mu_0 \vec{J}$ and $\vec{B} \equiv \nabla \times \vec{A}$ such that

$\nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{J}$.

We can rewrite the above equation using a vector identity,

$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$

What's nice about this form is it will let us make a simple decision about the divergence of $\vec{A}$ to simplify our problem. But first a reminder about V.

<u>Ambiguity of V (and by analogy $\vec{A}$)</u>

Recall that V was defined by $\nabla \times (-\nabla V) = 0$, such that we could always add a function φ to V whose gradient vanishes ($\nabla \varphi = 0$; a constant) and still have $\nabla \times (\nabla (V + \varphi)) = 0$.

That is we were free to choose the zero of V unless BCs dictated them.

The vector potential, $\vec{A}$, is defined by $\nabla\cdot(\nabla\times\vec{A})=0$ so we can always add a vector function, $\vec{\varphi}$, whose curl vanishes and still have,

$$\nabla\cdot(\nabla\times(\vec{A}+\vec{\varphi}))=0$$

- this is even more freedom than before because there's lots of interesting functions with zero curl!
- So with V, we often picked our "offset" (φ) to ensure that $V(\infty)\to 0$. this was an extra condition, a choice if you will, that we made to simplify our problem.
- So we will do the same thing here. We'll choose $\vec{\varphi}$ to modify $\vec{A}$ so it satisfies an extra restriction.

The Coulomb Gauge

A popular choice for magnetostatics is to pick $\vec{\varphi}$ such that $\nabla\cdot\vec{A}=0$ too.

→ we need to prove that we can always do this (proof later)

→ If we do this then $\nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}=\mu_0\vec{J}$

simplifies to $\boxed{\nabla^2\vec{A}=-\mu_0\vec{J}}$

which is really 3 Poisson equations!

$$\left.\begin{aligned}\nabla^2 A_x &= -\mu_0 J_x\\ \nabla^2 A_y &= -\mu_0 J_y\\ \nabla^2 A_z &= -\mu_0 J_z\end{aligned}\right\}$$

this kind of separation is only valid in Cartesian coords!

We know the solution to poisson's equation from working with V,

$$\nabla^2 V = -\rho/\epsilon_0 \longrightarrow V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')\,d\tau'}{\mathscr{r}}$$

so that means,

$$\nabla^2 A_x = -\mu_0 J_x \longrightarrow A_x = \frac{\mu_0}{4\pi}\int \frac{J_x(\vec{r}\,')\,d\tau'}{\mathscr{r}}$$

or that the sol'n for $\vec{A}$ in terms of $\vec{J}$ is,

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')\,d\tau'}{\mathscr{r}}}$$

$\vec{J}$ must be localized (like loops of current)

this choice of $\nabla\cdot\vec{A}=0$ is picking a gauge. In point of fact, it is picking the "Coulomb gauge."
Doing so helps deal with what are redundant degrees of freedom → more on this in 482

In any event, we should prove that we can always choose $\nabla\cdot\vec{A}=0$ if needed.

<u>Proof!</u> Suppose that $\nabla\times\vec{A}_0 = 0$ but $\nabla\cdot\vec{A}_0 \neq 0$

Let $\nabla\cdot\vec{A}_0 = f(\vec{r})$ some function

it's possible to find some function $\vec{\varphi}_0(\vec{r})$ such that,

$$\nabla\times(\vec{A}_0+\vec{\varphi}) = 0 \text{ and } \nabla\cdot(\vec{A}_0+\vec{\varphi}) = 0$$

We need

$$\nabla\times\vec{\varphi} = 0 \text{ to preserve } \nabla\times(\vec{A}_0+\vec{\varphi}) = 0 -$$

given that $\nabla\times\vec{A}_0 = 0$

and

$$\nabla\cdot\vec{\varphi} = -f(\vec{r}) \text{ to preserve } \nabla\cdot(\vec{A}_0+\vec{\varphi}) = 0$$

given that $\nabla\cdot\vec{A}_0 = f(\vec{r})$

We can guarentee that $\nabla\times\vec{\varphi} = 0$ if $\vec{\varphi} = \nabla\lambda$ for some scalar function $\lambda = \lambda(\vec{r})$

So because $\nabla \cdot \vec{\varphi} = -f(\vec{r})$,

$\nabla \cdot (\nabla \lambda(\vec{r})) = \nabla^2 \lambda = -f$ which is just Poisson's equation for λ. for any f you can solve this equation.

Thus,

for any f, λ exists, and then $\vec{A}_0 + \nabla \lambda$ gives $\vec{A}$ with the properties that $\nabla \times \vec{A} = \vec{B}$ and $\nabla \cdot \vec{A} = 0$.

(in fact there's an infinite # of λ's as you could always add a constant!)

Bottom line:

Given currents, $\vec{J}$, you have several choices to find $\vec{B}$,

1) Solve Ampere: $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$ (need good symmetries)

2) Solve Biot Savart: $\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times \hat{\mathscr{r}}}{\mathscr{r}^2} d\tau'$

or

3) Find $\vec{A}$ $\quad \vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')d\tau'}{\mathscr{r}}$ ← sometimes easier than Biot Savart no cross product!

and then take $\nabla \times \vec{A} = \vec{B}$. ← only good if $\vec{J}$ is localized.

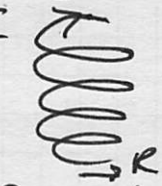
could also,

4) find (guess?) $\vec{A}$ that satisfies

$\nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{J}$ (and $\nabla \cdot \vec{A} = 0$ if you like)

then $\nabla \times \vec{A}$ gives $\vec{B}$.

Let's get some practice with $\vec{A}$.



Example 1: the infinite solenoid

Consider a solenoid with current I, radius R, and n turns / length.

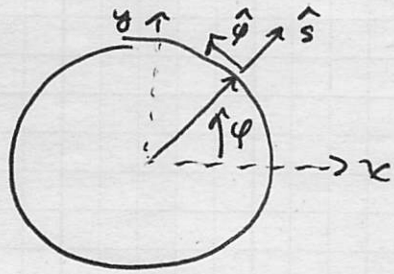
We found $\vec{B}$ using Ampere's Law, let's find $\vec{A}$ and check it!

we want to solve,

$$\nabla^2 \vec{A} = -\mu_0 \vec{J} = -\mu_0 \underbrace{I n \delta(r-R)\,\hat{\varphi}}$$

Does this make sense to you?

This $\vec{J}$ is a surface current on the cylinder.



From this figure we have,

$$\hat{\varphi} = \langle -\sin\varphi, \cos\varphi, 0 \rangle$$

why are we doing this?

Because we can write down each component,

$$\nabla^2 A_x = +\mu_0 n I \sin\varphi\, \delta(r-R)$$

$$\nabla^2 A_y = -\mu_0 n I \cos\varphi\, \delta(r-R)$$

These are the same types of equations we dealt w/ in electrostatics where, $\nabla^2 V = \rho/\epsilon_0$

if $\rho \propto \sin\varphi\, \delta(r-R)$ – some surface charge on a cylinder.

You could use separation of variables in cylindrical coordinates to find A_x & A_y

but we are going to attempt something more clever and less math intensive.

Consider the solenoid again,

$\uparrow \hat{z}$ $\quad$ a radius

n turns/length.

I current

We want to solve $\nabla^2 \vec{A} = \mu_0 \vec{J}$ but let's set that aside for a moment.

→ Consider a totally different problem, the fat wire.

If $\vec{J} = \begin{cases} J_0 \hat{z} & s \leq a \\ 0 & s > a \end{cases}$ we found that,

$$\vec{B}(s) = \begin{cases} \mu_0 J_0 s/2 \, \hat{\varphi} & s \leq a \\ \dfrac{\mu_0 J_0 a^2}{2s} \hat{\varphi} & s > a \end{cases}$$

This magnetic field and current satisfy

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

for the solenoid, we know that

$$\vec{B} = \begin{cases} B_0 \hat{z} & s \leq a \\ 0 & s > a \end{cases} \qquad \text{with } B_0 = \mu_0 n I$$

and this magnetic field had better satisfy,

$$\nabla \times \vec{A} = \vec{B}$$

These problems have precisely the same mathematical structure!

So we can just write down $\vec{A}$!

$$\vec{A} = \begin{cases} \dfrac{B_0 s}{2} \hat{\varphi} & s \leq a \\ \dfrac{B_0 a^2}{2s} \hat{\varphi} & s > a \end{cases}$$

$\vec{A}$ looks a bit like the current in that it points azimuthally like $\vec{J}$.

Let's check that $\vec{A}$ works,

$$\vec{A} = \begin{cases} \frac{B_0 s}{2}\hat{\varphi} & s \le a \\ \frac{B_0 a^2}{2s}\hat{\varphi} & s > a \end{cases}$$

Let's check $\nabla \times \vec{A} = \vec{B}$

the curl in cylindrical coordinates, at least the relevant part is,

$$\underbrace{-\frac{\partial A_\varphi}{\partial z}}_{0}\hat{s} + \frac{1}{s}\frac{\partial}{\partial s}(sA_\varphi)\hat{z}$$

$A_\varphi(s)$ only in our case.

$s \le a$: $\frac{1}{s}\frac{\partial}{\partial s}\left(\frac{B_0 s^2}{2}\right)\hat{z} = \frac{1}{s}\left(\frac{\cancel{2}}{2} B_0 s\right)\hat{z} = B_0\hat{z}$ checks out!

$s > a$ $\frac{1}{s}\frac{\partial}{\partial s}\left(\frac{B_0 a^2}{2}\right) = 0$ checks out!

Notice $\vec{A} \neq 0$ outside but $\vec{B} = 0$ (QUANTUM consequences)

so $\nabla \times \vec{A} = \vec{B}$ as expected.

Let's check if $\nabla \cdot \vec{A} = 0$

The divergence in cylindrical coordinates, at least the relevant part is, $\frac{1}{s}\frac{\partial A_\varphi}{\partial \varphi}$

inside & outside $A_\varphi(s)$ $\therefore$ $\frac{\partial A_\varphi}{\partial \varphi} = 0$ so $\nabla \cdot \vec{A} = 0$ as desired.

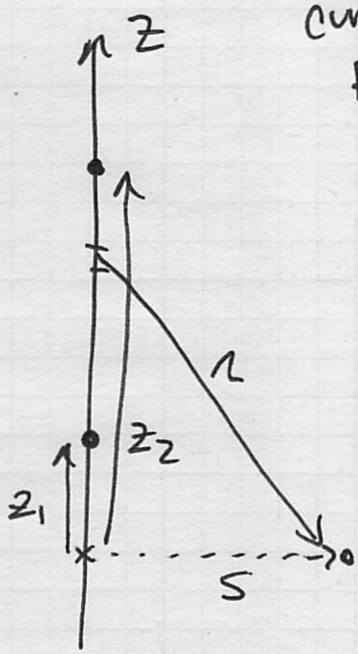
Physical interpretation of $\vec{A}$? $\nabla \times \vec{A} = \vec{B}$, so,

$$\oint \vec{A} \cdot d\vec{l} = \int (\nabla \times \vec{A}) \cdot d\vec{a} = \int \vec{B} \cdot d\vec{a} = \Phi_B \Leftarrow \text{magnetic flux!}$$

the circulation of $\vec{A}$ tells you the magnetic flux through the enclosed loop.

Here we used $\vec{B}$ to find $\vec{A}$. a bit silly, why find $\vec{A}$ when we know $\vec{B}$? just trying to visualize $\vec{A}$. for now, it seems to follow the current.

Example: segment of a long straight wire with current I.



Find $\vec{A}$.

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\,d\tau'}{\mathscr{r}}$$

$\vec{J} = I\,\delta(x')\,\delta(y')\,\hat{z} \rightarrow$ expect $\vec{A} = A\hat{z}$

$\vec{r} = \langle s, 0, 0\rangle \qquad \vec{r}' = \langle 0, 0, z'\rangle$

$\vec{\mathscr{r}} = \vec{r} - \vec{r}' = \langle s, 0, z'\rangle$

$\mathscr{r} = \sqrt{s^2 + z'^2} \qquad s^2 = x^2 + y^2$ technically,

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{I\,\delta(x')\,\delta(y')\,\hat{z}}{(s^2 + z'^2)^{1/2}}\,dx'dy'dz'$$

$$= \frac{\mu_0}{4\pi}\int_{z_1}^{z_2} \frac{I\,dz'\,\hat{z}}{(s^2 + z'^2)^{1/2}}$$

$$= \frac{\mu_0 I}{4\pi}\hat{z}\left[\ln\left(\sqrt{s^2 + z'^2} + z'\right)\right]_{z_1}^{z_2}$$

$$\vec{A} = \frac{\mu_0 I}{4\pi}\ln\left(\frac{\sqrt{s^2 + z_2^2} + z_2}{\sqrt{s^2 + z_1^2} + z_1}\right)\hat{z}$$

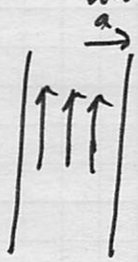
can find $\vec{B}$? $\nabla \times \vec{A}$ is the way

use cylindrical b/c expect $\vec{B} = B\hat{\phi}$, right?

Let's do a problem where we don't use $\vec{B}$ to get $\vec{A}$
But we will know the answer so we can check it.

Example 2: Infinite Wire

Consider an infinitely long wire of radius a with a uniform current I_0 running along it,



$\vec{I} = I_0\hat{z}$ so that $\vec{J} = I_0/\pi a^2\,\hat{z}$ $J_x, J_y = 0$
(inside the wire)

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\,d\tau'}{\mathscr{r}} \Rightarrow \text{consider each component,}$$

$$A_x = \frac{\mu_0}{4\pi}\int \frac{J_x\,d\tau'}{\mathscr{r}} = 0 \quad \text{so is } A_y = 0$$

for A_z, we could do the integral,

$$A_z = \frac{\mu_0}{4\pi}\int \frac{1}{\mathscr{r}}\,\frac{I_0}{\pi a^2}\,\hat{z}\,d\tau'$$

But let's consider the PDE,

$$\nabla^2 A_z = -\mu_0 J_z$$

J_z is a constant out to a and uniform in z

We have seen this equation before – think of a long wire with uniform λ, (or ρ): $\nabla^2 V = -\rho/\epsilon_0$ ← constant out to a; uniform in z.

We know the solution to this ($\nabla^2 V = -\rho/\epsilon_0$ with λ as discussed) for $\vec{E}$: Gauss' Law gives $E_{outside} = \dfrac{Q_{enc}/\epsilon_0}{2\pi s L}$

Let's just stay outside for now: $E_{outside} = \dfrac{\lambda}{2\pi\epsilon_0 s}$

$$V = -\int \vec{E}\cdot d\vec{l} = \frac{-\lambda}{2\pi\epsilon_0}\ln s = \frac{-\rho\pi a^2}{2\pi\epsilon_0}\ln s = \frac{-\rho a^2}{2\epsilon_0}\ln s$$

where ρ is the density of charge inside the wire.

This solution lets us read off A_z,

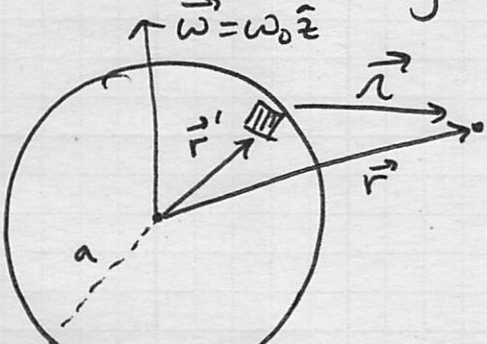
$$A_z = -\frac{\mu_0 a^2 J_z}{2}\ln s = -\frac{\mu_0 I_0}{2\pi}\ln s$$

Check! $\vec{B} = \nabla\times\vec{A} = -\dfrac{\partial A_z}{\partial s}\hat{\varphi} = +\dfrac{\mu_0 I_0}{2\pi s}\hat{\varphi}$ <u>as expected</u>

let's work on a slightly more complex example, the rotating sphere, which is done in Griffiths.
On our homework you found $\vec{K}$.

Example 3: Rotating spherical shell of charge

- Griffith's does this example, but we will do it a bit differently,



$$\vec{K} = \sigma \vec{v} = \sigma \vec{\omega} \times \vec{r}'$$

(recall $\vec{v} = \vec{\omega} \times \vec{r}'$)

With this $\vec{K}$, we want to compute,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r}')\,da'}{\imath}$$

Because this integral is over the surface of the sphere we find,

$$\vec{A}(\vec{r}) = \frac{\mu_0 \sigma}{4\pi} \iint_{sphere} \frac{(\vec{\omega} \times \vec{r}')\; a^2 \sin\theta'\, d\theta'\, d\varphi'}{\imath}$$

We can write this integral in the following way,

$$\vec{A}(\vec{r}) = \frac{\mu_0 \sigma a^2}{4\pi} \vec{\omega} \times \vec{f}(\vec{r}) \quad \text{where}$$

$$\vec{f}(\vec{r}) = \iint_{sphere} \frac{\vec{r}' \sin\theta'\, d\theta'\, d\varphi'}{\imath}$$

[Notice! $\vec{A}$ depends on $\vec{r}$ and thus $\vec{f}$ depends on $\vec{r}$. There is no other vector in this integral and thus $\vec{f}(\vec{r}) = C\vec{r}$ where C can itself depend on $\vec{r}$, but we expect $\vec{f}$ to point in $\hat{r}$]

We will do this by considering a particular direction $\vec{r} = \langle 0, 0, z \rangle$ and realize that in general ~~we~~ we get something quite similar.

Outside: $r>a$

Let's choose $\vec{r} = \langle 0,0,z\rangle$ outside the sphere

Then we expect $\vec{f}(\vec{r}) = c\langle 0,0,z\rangle = cz\hat{z}$

or $f_z = cz$ Let's check what f_z becomes

$$f_z = \iint \frac{z'\sin\theta' d\theta' d\varphi'}{\mathscr{r}}$$

$$f_z = \iint \frac{z'\sin\theta' d\theta' d\varphi'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \quad \left(\begin{array}{c} z' = a\cos\theta' \\ \text{projection} \\ \text{on } z' \text{ axis}\end{array}\right)$$

$$f_z = \int_0^{2\pi}\int_0^{\pi} \frac{a\cos\theta' \sin\theta' d\theta' d\varphi'}{\sqrt{a^2+z^2-2zz'}} \leftarrow \left(x'^2+y'^2+z'^2 = a^2 \text{ \& expanding } (z-z')^2\right)$$

$$f_z = 2\pi a \int_0^{\pi} \frac{\cos\theta'\sin\theta' d\theta'}{\sqrt{a^2+z^2-2za\cos\theta'}} \leftarrow (\text{again } z' = a\cos\theta')$$

How to integrate this? u sub, $u = \cos\theta'$ $du = -\sin\theta' d\theta'$

$$f_z = 2\pi a \int_{-1}^{+1} \frac{u\,du}{\sqrt{a^2+z^2-2azu}} = 2\pi a \frac{2}{3}\frac{a}{z^2}$$

So $f_z = cz = \frac{4}{3}\pi \frac{a^2}{z^2}$ In this case, we chose $\vec{r} = \langle 0,0,z\rangle$.

If we had chose $\langle x,0,0\rangle$ or $\langle 0,y,0\rangle$ we'd have obtained similar expressions for f_x and f_y such that,

$$\vec{f}(\vec{r}) = \frac{4}{3}\frac{\pi a^2}{r^3}\vec{r} \quad \text{for } r>a$$

$$\vec{A}(\vec{r}) = \frac{\mu_0 \sigma a^2}{4\pi}\vec{\omega}\times\vec{f}(\vec{r}) = \frac{\mu_0\sigma a^4}{3r^3}\vec{\omega}\times\vec{r}$$

for $\vec{r}$ outside the sphere

Inside: $r<a$

Inside $z<a$, so the integral gives $f_z = 2\pi a \frac{2}{3}\frac{z}{a^2}$ instead.

$cz = \frac{4\pi}{3a} z \Rightarrow c = \frac{4\pi}{3a}$ thus, $\vec{f}(\vec{r}) = \frac{4\pi}{3a}\vec{r}$

$$\vec{A}(\vec{r}) = \frac{\mu_0\sigma a^2}{4\pi}\vec{\omega}\times\vec{f}(\vec{r}) = \frac{\mu_0\sigma a}{3}\vec{\omega}\times\vec{r} \quad \text{for } r<a$$

the bottom line is that,

$$\vec{A}(\vec{r}) = \tilde{c}\,\vec{\omega}\times\vec{r}$$

where $\tilde{c}$ is given on the previous page, is a function of r, and differs for $r<a$ & $r>a$.

$\vec{\omega}\times\vec{r} = \omega r \sin\varphi\,\hat{\varphi}$ ($\vec{A}$ "follows" the current.)

<u>What about $\vec{B}$?</u>

$$\vec{B} = \nabla\times\vec{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, A_\varphi)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi)\hat{\theta}$$

this distribution produces a magnetic dipole, which will investigate later.

Now that we've worked through a few examples of $\vec{A}$, let's revisit Boundary Conditions ~~of~~ on $\vec{B}$ and then $\vec{A}$.

<u>Boundary Conditions</u>

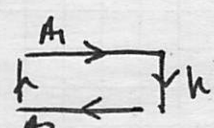
Like the electric field and the electric potential, $\vec{B}$ also has simple behaviors at surfaces (and boundaries). We've seen them before.

- $\nabla\cdot\vec{B}=0$ tells us that the perpendicular component of $\vec{B}$ is continuous. $B^{\perp}_{above} - B^{\perp}_{below} = 0$

- $\nabla\times\vec{B}=\mu_0\vec{J}$ or $\oint\vec{B}\cdot d\vec{l}=\mu_0 I_{enc}$ tells us that the parallel component of $\vec{B}$ is continuous unless there's a surface current at the boundary.

$$B^{\parallel}_{above} - B^{\parallel}_{below} = -\mu_0 K$$

for $\vec{A}$, $\oint\vec{A}\cdot d\vec{l} = \Phi_{magnetic}$ if Φ is finite



$A_1 L - A_2 L \to 0$ as $h\to 0$; $\Phi\to 0$. $\vec{A}$ is continuous (like V!)

Summary of $\vec{J}$, $\vec{A}$, & $\vec{B}$.

$$\begin{tikzcd}[column sep=small] & \vec{J} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \vec{A} \arrow[rr, leftrightarrow] & & \vec{B} \end{tikzcd}
\quad \text{is much like} \quad
\begin{tikzcd}[column sep=small] & \rho \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ V \arrow[rr, leftrightarrow] & & \vec{E} \end{tikzcd}$$

- Given $\vec{J}$, we can find $\vec{B}$ (Biot-Savart Law)

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,') \times \vec{\scriptstyle{r}}}{\scriptstyle{r}^3}$$

- Given $\vec{B}$, we can find $\vec{J}$ (Ampere's Law)

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

- Given $\vec{A}$, we can find $\vec{B}$ (definition with $\vec{A}$)

$$\nabla \times \vec{A} = \vec{B}$$

- Given $\vec{B}$, we can find $\vec{A}$ (haven't really done this except in special cases)
- Given $\vec{A}$, we can find $\vec{J}$ (easy when $\nabla \cdot \vec{A} = 0$)

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

- Given $\vec{J}$, we can find $\vec{A}$ (direct integration)

$$\vec{A} = +\frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,')\, d\tau'}{\scriptstyle{r}}$$

The Boundary Conditions of $\vec{A}$ & $\vec{B}$ will come in handy w/ materials (currents at boundaries)

- Much like $\vec{E}$ in materials with free charges.