- So far we developed techniques that allow us to solve (some restricted set of) problems exactly. We've developed ways of find the electric potential, which leads us to the electric field by virtue of the gradient.

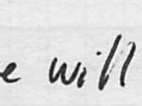
$$\left.\begin{aligned} &V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')\,d\tau'}{\mathscr{r}'} \\ \text{and}\quad &\text{Solve } \nabla^2 V = 0 \end{aligned}\right\} \vec{E} = -\nabla V$$

- But sometimes an approximate answer will do just fine because it captures much of the essential physics. Morever, a greater swath of problems can be understood using some form of approximation than can be solved completely analytically.
- An incredibly powerful and far-reaching technique is called the <u>Multipole Expansion</u>. This expansion and methods like it are ubiquitous in theoretical physics.

To begin our study of this technique, we will start with understanding the physics of two charges in a new way: from the perspective of the "dipole" (<u>two</u> po<u>les</u>)

<u>The electric Dipole</u>

The <u>physical</u> electric dipole consists of two equal magnitude, oppositely signed charges (q & $-q$) separated by d.



0 net <u>charge</u>

For a general charge configuration, we will define a dipole moment (we will see why soon) to be,

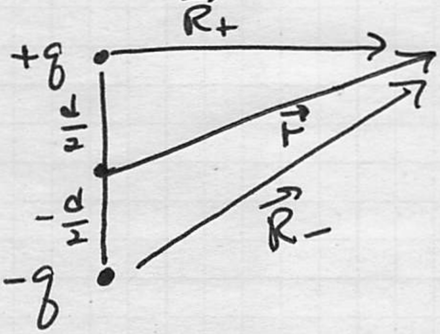
$$\vec{p} = \sum_i q_i \vec{r}_i$$

For the configuration shown on the previous page, $\vec{p} = q\vec{d}$ Clicker Question: more charges, $\vec{p}$?

Electric Potential & Field of a physical Dipole

For the charge configuration of a physical dipole, we can derive the approximate potential & E-field far from the dipole.

Through doing this we will see why it makes sense to define a dipole moment, $\vec{p} = q\vec{d}$.



$\langle x, y, z \rangle$ Let's find $V(\vec{r})$,

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right)$$

We can write R_+ & R_- down from the figure to the left:

$$R_+ = \sqrt{x^2+y^2+(z-d/2)^2} \quad R_- = \sqrt{x^2+y^2+(z+d/2)^2}$$

so,

$$R_\pm = \sqrt{x^2+y^2+\left(z \mp \frac{d}{2}\right)^2}$$

We can take out a z^2 b/c $x^2+y^2+z^2=r^2$,

$$R_\pm = \sqrt{x^2+y^2+z^2\left(1 \mp \frac{d}{2z}\right)^2}$$

$$= \sqrt{x^2+y^2+z^2\left(1 \mp \frac{d}{z} + \frac{d^2}{4z^2}\right)}$$

$$= \sqrt{(x^2+y^2+z^2) \mp dz + d^2/4}$$

$$R_\pm = \sqrt{r^2 \mp dz + d^2/4}$$

If we are interested in approximate solutions where $d \ll r$. that is, we are far from the dipole, then,

$$R_{\pm} = \sqrt{r^2 \mp dz + d^2/4} = r\sqrt{1 \mp \frac{dz}{r^2} + d^2/4r^2}$$
$$\approx r\left(1 \mp \tfrac{1}{2}\,dz/r^2\right)$$

We go back to $V(\vec{r})$ and find,

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right)$$

here we used: $\frac{1}{(1\pm\varepsilon)} \approx (1\mp\varepsilon)$ $\varepsilon \ll 1$

$$\approx \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r}\left(1+\frac{1}{2}\frac{dz}{r^2}\right) - \frac{1}{r}\left(1-\frac{1}{2}\frac{dz}{r^2}\right)\right)$$
$$\approx \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{r} + \frac{1}{2}\frac{dz}{r^3} + \frac{1}{2}\frac{dz}{r^3}\right)$$
$$V(\vec{r}) \cong \frac{q}{4\pi\epsilon_0}\frac{dz}{r^3} = \frac{qdz}{4\pi\epsilon_0 r^3}$$

(Notice $p = qd$ is buried in there)

From the figure above, $\vec{p}\cdot\vec{r} = p_z r_z = qdz$!
So a slightly more general form of this approximate result for $V(\vec{r})$ is,

$$V(\vec{r}) \approx \frac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_0 r^3} = \frac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2}$$

Notice that this potential is not a Coulomb one!
It drops off like $1/r^2$ not $1/r$!

Also notice that the result is "coordinate free".
- We can choose any axes we like and still describe the dipole.

<u>What does the electric field look like?</u>

$\vec{E}(\vec{r}) = -\nabla(V(\vec{r}))$, we could do this "coordinate free", but instead let's choose spherical coordinates: $V(\vec{r}) = \dfrac{p\cos\theta}{4\pi\epsilon_0 r^2}$

$$\vec{E}(\vec{r}) = -\nabla V(\vec{r}) = -\frac{\partial V}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta}$$

$$E_r = -\frac{\partial V}{\partial r} = +\frac{2p\cos\theta}{4\pi\epsilon_0 r^3}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3}$$

so, $\vec{E} = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$

Recall this is $r \gg d$ so that it is an approximate form of the Electric Field.

<u>Clicker Question</u>: What's $\vec{E}(\vec{r}=0)$?

There's an important distinction here we should make about real, physical dipoles & idealized ones.

<u>Real vs. Ideal Dipoles</u>

A real dipole is two charges $(+q, -q)$ seperated by d. They can be characterized by a dipole moment, $\vec{p} = q\vec{d}$. But they are <u>not</u> pure dipoles. A pure dipole produces <u>only</u> a dipole field. In that case $d \to 0$ as $q \to \infty$ such that p stays finite.

Let's look at the structure of the field.

At $\theta = \pi/2$ $\quad \vec{E} = \frac{p}{4\pi\epsilon_0 r^3}\hat{\theta}$ points "down"

$\theta = 0$ $\quad \vec{E} = \frac{p}{4\pi\epsilon_0 r^3} 2\hat{r}$ points "up"

<u>Ideal</u>

It's a pointlike dipole

<u>Real</u> (looks similar far away but....)

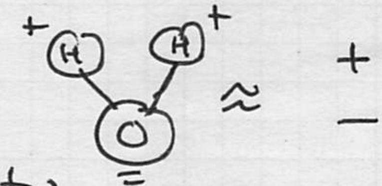
up close it's not right, we will need more terms!

Where do dipoles come from?

1) There are natural dipoles that are the result of polar molecules.

For example, water molecule

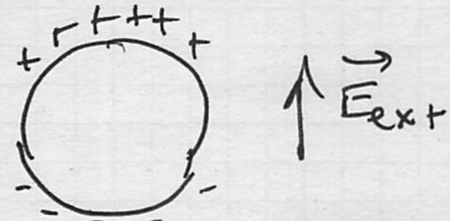
(Dipoles are very important in Biophysics!!)



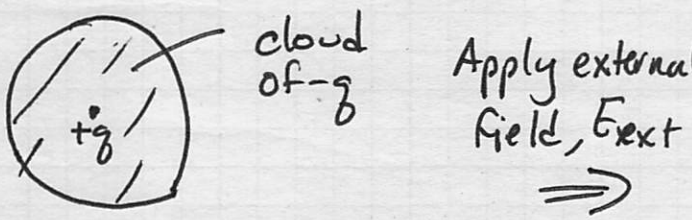
+ Typically, "d" is on the order of Angstroms $\sim 10^{-10}$ m and "q" is about one electron charge $\sim 10^{-19}$ C, so the dipole moments have a roughly $p \simeq 10^{-28}$ Cm

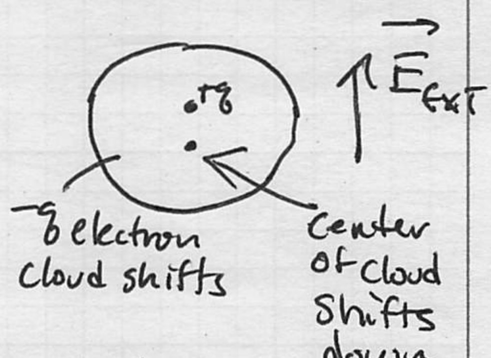
2) We can easily create induced dipoles by putting neutral objects into electric fields and polarizing them.

this can happen to atoms, too!



Think of the typical simple cloud model of the atom with the dense nucleus.

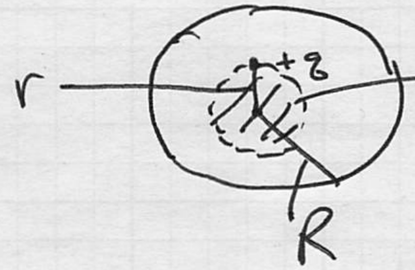




+ So at equilibrium there's a pull on the $+q$ towards the $-q$ charge center, which is equal to the push on it by the external field.

Note! the force $+q$ experiences to the cloud is NOT $q^2/4\pi\epsilon_0 d^2$ b/c there's negative charge smeared out!

We need the field at the location of the +q charge due sphere inside d.



minus charge contributing to $\vec{F}$ on +q.

Using Gauss' Law that field is $E = \frac{+qr}{4\pi\epsilon_0 R^3}$ where $r=d$.

so

$$F_{\substack{\text{on } +q \\ \text{due to} \\ \text{cloud}}} = F_{\substack{\text{on } +q \\ \text{due} \\ \text{to } E_{ext}}},$$

$$\frac{q^2 d}{4\pi\epsilon_0 R^3} = q E_{EXT} \Rightarrow qd = p = 4\pi\epsilon_0 R^3 E_{EXT}$$

+ We find that the dipole moment is linear in the External Field, E_{ext} (linear polarization)
+ We can estimate the "polarizability" of an atom with this model, $P/E \equiv \alpha \simeq 4\pi\epsilon_0 R^3$ ← size of atom $R \sim 10^{-10}$ m

We will come back to this when we get to E fields in matter.

<u>Let's</u> go <u>back to physical dipoles for a moment</u>

the general prescription for the dipole moment is $\vec{p} = \sum_i q_i \vec{r}_i$. ~~Does~~ Is this moment a property of the charges or does it contain other info?

<u>Clicker!</u>

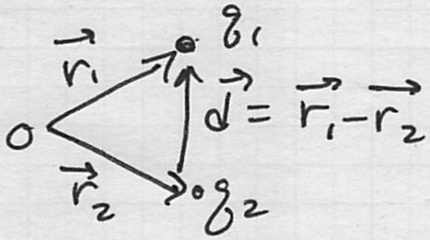
Consider d ↓ $\cdot +2q$ ↑ $\cdot -2q$ what's p?

What if the charges aren't the same magnitude? $+2q \cdot$ $-q \cdot$ ← origin vs. $\cdot +2q$ • ← origin $\cdot -q$

<u>Clickers!</u>

Only if the total charge of the dipole is zero is the dipole moment independent of the location of the dipole! (so the "monopole" moment must vanish!)

Formal Proof:



$$\vec{p} = \sum_i q_i \vec{r}_i = q_1 \vec{r}_1 + q_2 \vec{r}_2$$

so $\vec{p}$ is necessarily grounded in $\vec{r}_1 + \vec{r}_2$ unless,

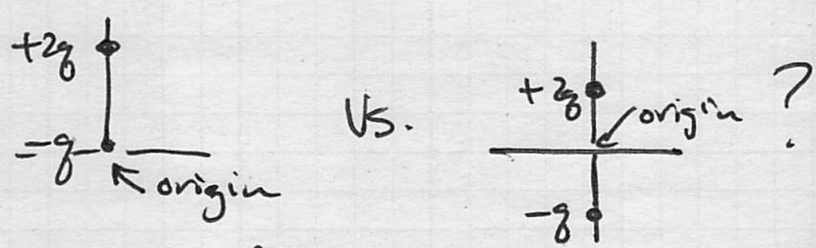
$q_1 = -q_2$ then,

there's the rub.

$$\vec{p} = q_1 \vec{r}_1 - q_1 \vec{r}_2 = q_1 (\vec{r}_1 - \vec{r}_2) = q\vec{d}$$

$\vec{d}$ is indepent of coordinate system origin, it's the relative location of q_1 w.r.t. q_2.

So what's the deal with



Well, the higher order terms are need to give us the full details

+ This means for general charge configurations we will to know something about other terms in the multipole expansion.
+ Also a corrollary to our finding above is that the lowest non-vanishing term in the multipole expansion will be independent of the choice of origin!

So let's explore this expansion and see what we make sense of along the way.

Multipole Expansion

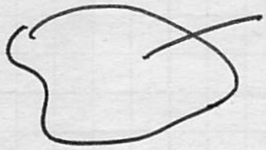
We know a bit about the potential now for a point charge, q, and a dipole, $\vec{p}$.

$$V_q(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \longleftrightarrow \text{dies off like } 1/r \quad \left(\text{looks the same at every } \theta\right)$$

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} \longleftarrow \text{dies off like } 1/r^2 \quad \left(\vec{p}\cdot\hat{r} = p\cos\theta \text{ so there is angular dep.}\right)$$

Clicker Question(s): When can you use $\vec{p}\cdot\hat{r}$ vs. $\sum q_i \vec{r}_i$?

Electric monopoles are single charges. In our case, we can think of so distribution as having total charge Q,



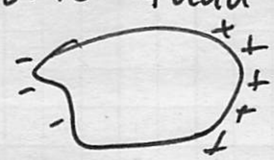
Q_{tot}

so far away the potential will have a piece that dies off like $\sim 1/r$

Why? $\Longleftarrow$

B/c far enough away the blob there looks like a pt. charge.

Electric dipoles (real ones) are separated (polarized) charges. In our case, maybe the blob has a bit more of the charges making up Q on on side than on the other,



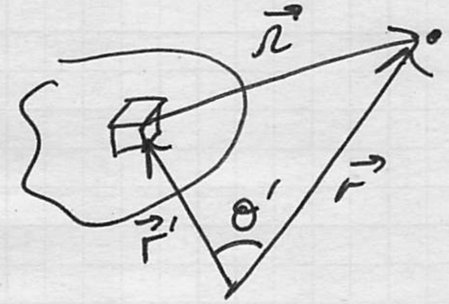
still Q total but more + on right

So if we want a bit more detail, there's a small $1/r^2$ field (dipole field) far away as well

This conceptual argument suggests there's a method of approximation (or estimation) we can develop → this is the Multipole expansion!

Let's derive this result in detail for any arbitrary distribution. So we go back to,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r}\,')\,d\tau'}{\mathscr{r}}$$



Conceptually: If $\vec{r}$ is very far from the distribution then $\rho(\vec{r}\,')$ will look simple.

(like a pt charge if $q_{net} \neq 0$; like a dipole if $q_{net} = 0$ like a quadrapole if all the dipoles cancel $\begin{smallmatrix}+&-\\-&+\end{smallmatrix}$)

Let's do the math to find how this works out.

from figure: $\mathscr{r}^2 = r^2 + r'^2 - 2rr'\cos\theta' = r^2\left(1 + \frac{r'^2}{r^2} - \frac{2r'}{r}\cos\theta\right)$

we pull out r^2 b/c far away it will be large compared to r', so we can approximate $(r'/r \ll 1)$

Lets say $\varepsilon \equiv r'/r \ll 1$ when we are far away.

$\mathscr{r}^2 = r^2(1 - 2\varepsilon\cos\theta' + \varepsilon^2)$ so,

$\mathscr{r} = r\sqrt{1 - 2\varepsilon\cos\theta' + \varepsilon^2}$ (for this we will need $1/\mathscr{r}$ instead)

We will expand this in orders of ε as it should converge if $\varepsilon \ll 1$.

Recall that

$$(1+\eta)^{-1/2} = 1 + \frac{(-1/2)}{1!}\eta + (-\tfrac{1}{2})(-\tfrac{1}{2}-1)\frac{\eta^2}{2!} + \cdots$$

With,

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left(1 - 2\varepsilon\cos\theta' + \varepsilon^2\right)^{-1/2}$$

$$\frac{1}{\mathscr{r}} \simeq \frac{1}{r}\left(1 - \frac{1}{2}\left(-2\varepsilon\cos\theta' + \varepsilon^2\right) + \left(\frac{3}{8}\right)\left(-2\varepsilon\cos\theta' + \varepsilon^2\right)^2 + \ldots\right.$$

We can collect all terms in order of ε, ε^2, ~~and so on~~

$$\frac{1}{\mathscr{r}} \simeq \frac{1}{r}\left(1 + \varepsilon\cos\theta' - \frac{1}{2}\varepsilon^2 + \frac{3}{8}4\varepsilon^2\cos^2\theta' + \underbrace{O(\varepsilon^3)}_{\text{higher terms.}}\right)$$

$$\frac{1}{\mathscr{r}} \approx \frac{1}{r}\left(1 + \varepsilon\cos\theta' + \left(-\frac{1}{2} + \frac{3}{2}\cos^2\theta'\right)\varepsilon^2 + \ldots\right)$$

So we find that $V(\vec{r})$ can be approximated,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r}\,')d\tau'}{r}\left(1 + \varepsilon\cos\theta' + \left(-\frac{1}{2} + \frac{3}{2}\cos^2\theta'\right)\varepsilon^2 + \ldots\right)$$

Because $\varepsilon \ll 1$ — leading term — smaller — even smaller!

Because the terms are additive, we can look at ~~them~~ one at a time.

<u>Monopole</u> <u>Term</u>

The leading term we will find to be the "monopole term" – it will only care about the net charge. Let's see how,

$$V(\vec{r}) \approx \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r}\,')d\tau'}{r} = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\underbrace{\int_V \rho(\vec{r}\,')d\tau'}_{Q_{tot}!}$$

constant, <u>not</u> <u>primed</u>

So, $V(\vec{r}) \cong \dfrac{Q_{tot}}{4\pi\epsilon_0 r}$

just the potential of a pt charge a distance r from the observation location.

Dipole Term

The next term we will find will tell us about simple (linear) asymmetries in our charge distribution. The integral will give the dipole moment. Let's see how,

$V(\vec{r})$ contains this integral,

$$\frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r}\,')d\tau'}{r}\,\varepsilon\cos\theta' \quad \text{where } \varepsilon = r'/r$$

$$= \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r}\,')d\tau'}{r}\,\frac{r'}{r}\cos\theta'$$

$$= \underbrace{\frac{1}{4\pi\epsilon_0 r^2}}\underbrace{\int_V \rho(\vec{r}\,')\,r'\,d\tau'\cos\theta'}$$

notice that this term will drop off faster than the leading term (dies off like $1/r^2$)

this integral is just a number that is independent of r. It is a property of the distribution! Turns out it's the dipole moment!

Also note that $\cos\theta' = P_1(\cos\theta')$ — the 1st Legendre Poly (curious?)

Quadrupole Term

The next term will tell us about more complex asymmetries in the distribution. The integral will be the "quadrupole moment" of the distribution. Let's see,

the next term is,

$$\frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r}\,')d\tau'}{r}\left(-\frac{1}{2}+\frac{3}{2}\cos\theta'\right)\varepsilon^2 \quad \text{where } \varepsilon^2 = \frac{r'^2}{r^2}$$

$$= \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r}\,')d\tau'}{r}\,\frac{r'^2}{r^2}\underbrace{\left(-\frac{1}{2}+\frac{3}{2}\cos\theta'\right)}_{\text{this is } P_2(\cos\theta')!}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\underbrace{\int_V \rho(\vec{r}\,')\,r'^2\,d\tau'\,P_2(\cos\theta')}_{\text{Quadrupole moment.}}$$

which dies off faster still!

So our Expansion gives us,

$$V(\vec{r}) \simeq \frac{1}{4\pi\epsilon_0}\left(\frac{Q_{tot}}{r} + \frac{\text{dipole moment}}{r^2} + \frac{\text{quadrupole moment}}{r^3} + \dots\right)$$

When r is big the leading order term will dominate V and thus $\vec{E}$ and thus the physics.*

*Note: many times in physics we care about the nuance so just b/c these terms might be small doesn't we won't care about them!

So if $Q_{tot} = 0$ then dipole term will dominate.
if $Q_{tot} = 0$ & $p = 0$ the quadrupole term dominates
etc.

Typically, that dipole term will be our first correction ($Q_{tot} \neq 0$) or first non-zero term ($Q_{tot} = 0$), so it's worth unpacking it a bit more.

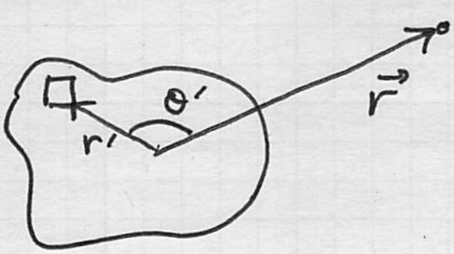
Let's imagine $Q_{tot} = 0$ for now,

<u>Dipole Term (with $Q_{tot} = 0$ so leading term)</u>

$$V_{dip}(r) \cong \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int_V \rho(\vec{r}\,')\, d\tau'\, r'\cos\theta'$$

what is $r'\cos\theta'$?

$r'\cos\theta' = \vec{r}\,' \cdot \hat{r}$ from the figure



$$V_{dip}(r) \simeq \frac{1}{4\pi\epsilon_0 r^2}\int_V \rho(\vec{r}\,')\,\vec{r}\,'\cdot\hat{r}\, d\tau'$$

$$\simeq \frac{1}{4\pi\epsilon_0 r^2}\left(\int_V \rho(\vec{r}\,')\,\vec{r}\,'\, d\tau'\right)\cdot\hat{r}$$

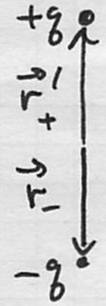
Recall that for a pure dipole, $V = \frac{1}{4\pi\epsilon_0 r^2}\vec{p}\cdot\hat{r}$

In general, $\vec{p}$ = "dipole moment" $\equiv \int_V \rho(\vec{r}\,')\,\vec{r}\,'\, d\tau'$

The dipole moment

$$\vec{p} \equiv \int_V \rho(\vec{r}\,')\,\vec{r}\,'\,d\tau'$$ this is the general definition (recall $\vec{p} = \sum q_i \vec{r}_i$?)

For point charges, we find,



$$\vec{p} = \int_V q\,\delta^3(\vec{r}\,'_+)\,\vec{r}\,'d\tau' + \int_V -q\,\delta^3(\vec{r}\,'_-)\,\vec{r}\,'d\tau'$$

$$= q\vec{r}\,'_+ + (-q)\vec{r}\,'_-$$ which is $\sum_i q_i \vec{r}_i$

Clicker Questions: q/r^2 ones.

Clicker Questions: $V(r) = \ldots$

We can compute the dipole moment for *any* distribution

Example:

Consider a spherical shell with $\rho = \sigma_0\,\delta(r-R)\cos\theta$

We expect it to have a dipole moment because,

The ϕ integral kills off x + y

$$P_z = \int_V \rho(\vec{r}\,')\,z'\,d\tau'$$

$$= \int_V \sigma_0\,\delta(r'-R)\cos\theta'\,r'\cos\theta'\,r'^2\sin\theta'\,d\theta'\,d\phi'\,dr'$$

$$= \sigma_0 \int_0^\infty \delta(r'-R)\,r'^3\,dr' \int_0^{2\pi} d\phi' \int_0^\pi \cos^2\theta'\sin\theta'\,d\theta'$$

$$= 2\pi\sigma_0 R^3 \int_0^\pi \cos^2\theta'\sin\theta'\,d\theta'$$

$$= 2\pi\sigma_0 R^3 \left(-\frac{\cos^3\theta}{3}\right)\Big|_0^\pi = \frac{4\pi}{3}\sigma_0 R^3$$

Clicker Question: Different distributions.