Electromagnetism deals w/ electric & magnetic interactions. In 184 you learned quite a bit about these interactions and the fields that cause them. Your concept maps demonstrated that you have a sense of how these things connect together.

Sample Map

Concept of a Field

$\vec{E}$ ← sources: Q, later $d\Phi_B/dt$ — Gauss & Faraday's Laws

$\vec{E}$ → effects: $\vec{F} = q\vec{E}$

$\vec{B}$ ← sources: I, later $d\Phi_E/dt$ — Ampere's Law

$\vec{B}$ → effects: $\vec{F} = q\vec{v}\times\vec{B}$

From $\vec{E}$ → Voltage (Potential)
Work & Energy

$\vec{E}$ & $\vec{B}$ in matter (Conductors)

In 481 (& 482), we will extend this understanding to include:

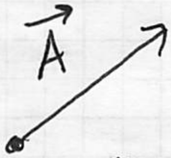
- Applying vector calculus (today!)
- Laplace's Eqn
- Vector Potential
- Approximation Techniques

} Solving for V ⇒ $\vec{E}$; later... Solving for $\vec{A}$ ⇒ $\vec{B}$

& Maxwell's Equations

Reminders about Vectors

I'm going to assume you are relatively familar with vectors and the Cartesian Coordinate system.



$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z} = \langle A_x, A_y, A_z \rangle$

DC shorthand

magnitude: $|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

unit vector: $\hat{A} = \dfrac{\vec{A}}{|\vec{A}|}$

in notes and hw, will use <u>bold face</u>

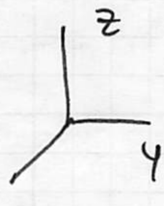
☆ Clicker Question: Direction of Force on Charge.

I'm also going to assume you are familar with "multiplying" vectors:

Scalar Product: $\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z = AB\cos\theta_{AB}$

Vector Product: $\vec{A}\times\vec{B} = \det\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & \cancel{B_y} A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$

☆ Clicker Questions:



$\vec{A}\times\vec{B}$?

$\vec{B}\times A$?

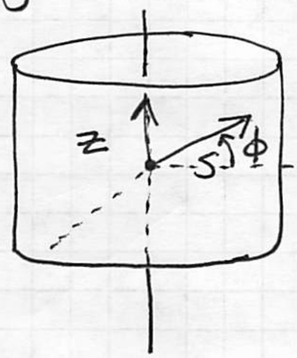
Can you say anything in general about

$\vec{A}\times\vec{B}$ & $\vec{B}\times\vec{A}$?

In 481, we are going to extend these kinds of operations (and, thus, your knowledge of vectors) to Cylindrical & spherical coordinate systems.

We will assume you have developed some knowledge of these systems, but we will spend a fair amount of time reviewing & using them.

Cylindrical Coordinates (as per Griffith's*)



$s: 0 \to \infty$

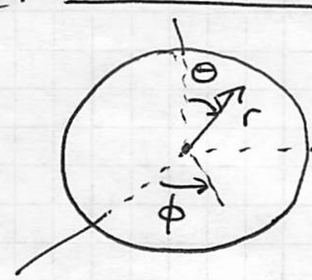
$z: -\infty \to \infty$

$\phi: 0 \to 2\pi$

* Be careful with other sources as symbols might change!

The front fly-leaf of Griffith's will become your best friend in this class

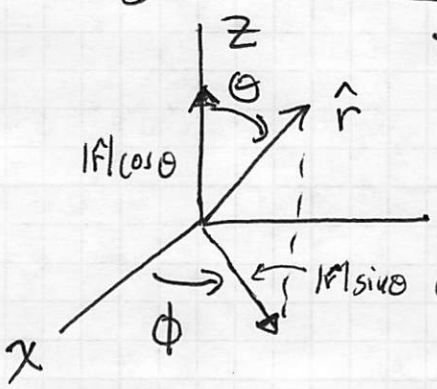
Spherical Coordinates (as per Griffith's*)



$r: 0 \to \infty$

$\left.\begin{array}{l}\phi: 0 \to 2\pi \\ \theta: 0 \to \pi\end{array}\right\}$ * probably different from your math class.

Activity: Derive $\hat{r}$



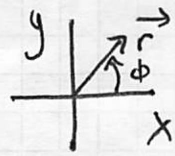
the tail of a vector doesn't matter unless we are defining a vector field, so place $\hat{r}$ at the origin to simplify.

- Easiest to pick off is $\hat{z}$, just $|\hat{r}|\cos\theta = \cos\theta$

- That means that $|\hat{r}|\sin\theta$ will project $\hat{r}$ into the x-y plane. $|\hat{r}|\sin\theta = \sin\theta$
- Then $\cos\phi$ gives projection along x and $\sin\phi$ along y so that

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

you can do this for $\hat{\theta}$ & $\hat{\phi}$, but fly-leaf has all of them.

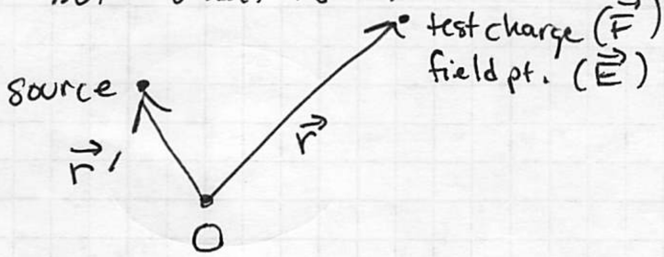
★ Clicker Question: 

what is $\vec{r}$ in cylindrical coordinates?

So why are we doing all this vector manipulation? B/c 481 is about extending our understanding of $\vec{E}$ & $\vec{B}$ beyond our 184 descriptions

For example, in 184: $F = \frac{k q_1 q_2}{r^2}$ & $E = \frac{k q_1}{r^2}$

★ but what is r^2?



★ Clicker Question: How is $\vec{R}_{12}$ related to $\vec{r}_1$ + $\vec{r}_2$?

Griffith's script-r, $\vec{\mathscr{r}}$

$\vec{\mathscr{r}}$ denotes the relative position vector between the source ($\vec{r}\,'$) and the field pt. ($\vec{r}$),

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' \quad \left(\text{observation location} - \text{source location}\right)$$

$\vec{\mathscr{r}}$ is needed to define $\vec{E}$ + $\vec{B}$, as you will see,

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r}-\vec{r}\,'|^2} \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|} = \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{\mathscr{r}}|^2} \hat{\mathscr{r}}$$

★ Clicker Question! Coulomb's Law & $\hat{\mathscr{r}}$

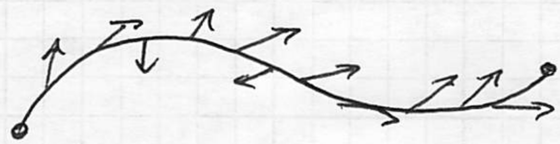
Script-r will be a real challenge as it contains two vectors, $\vec{r}$ & $\vec{r}\,'$. When thinking about $\vec{\mathscr{r}}$ remember that $\vec{r}$ points to the field pt. and $\vec{r}\,'$ pts to the source. This can often simplify things.

In addition to vector formalism, we will dust of our integration skills as much of advanced E&M is about adding the effects of continuous distributions of charge. Remember:

$$\sum \text{(discrete)} \quad \text{vs.} \quad \int \text{(continuous)}$$

Line Integrals

$$\int \vec{V} \cdot d\vec{l}$$



Example: Compute Work done by $\vec{F} = \langle a, x \rangle$ along the line $y = 2x$ from $\langle 0,0 \rangle$ to $\langle 1,2 \rangle$.

$$W = \int_P \vec{F} \cdot d\vec{l}$$

along the path → P

☆ Clicker Question: What is $d\vec{l}$?

In general, $d\vec{l} = \langle dx, dy, dz \rangle$; as this is only 2D $d\vec{l} = \langle dx, dy \rangle$.

We can form the Integral now,

$$W = \int_P \langle a, x \rangle \cdot \langle dx, dy \rangle$$

☆ Clicker Question: Which integral form is correct?

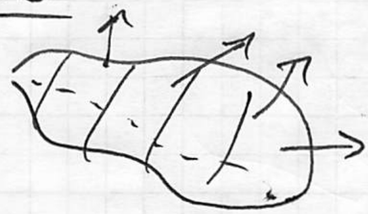
it's an integral along an established path, so it should be a 1D integral along that path. We can use $y = 2x \rightarrow dy = 2dx$ to cast the integral that way,

$$W = \int_P a\,dx + x\,dy = \int_P a\,dx + 2x\,dx$$

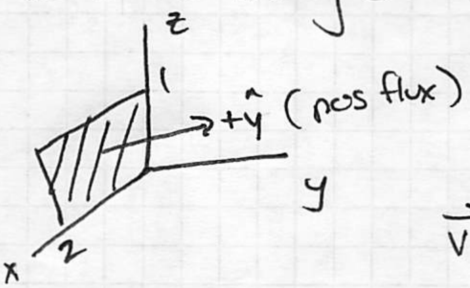
$$= \int_0^1 (a + 2x)\,dx = ax + x^2\Big|_0^1 = a + 1$$

Surface Integrals

$$\int_S \vec{v}\cdot d\vec{A}$$



Example: A fluid with velocity, $\vec{v} = \langle x, b, 0\rangle$ flows in some region of space. What is the fluid flux through the x-z plane bounded by 0 & 2 along x and 0 & 1 along z. Assume $+\hat{y}$ is positive flux.



* CQ: Which component(s) give rise to the fluid flux?

$\vec{v} = \langle x, b, 0\rangle \quad d\vec{A} = \langle 0, dxdz, 0\rangle$

$$\int_S \vec{v}\cdot d\vec{A} = \int_S \langle x, b, 0\rangle \cdot \langle 0, dxdz, 0\rangle$$

$$= \int_S b\,dxdz = b\int_0^1 dz \int_0^2 dx = 2b$$

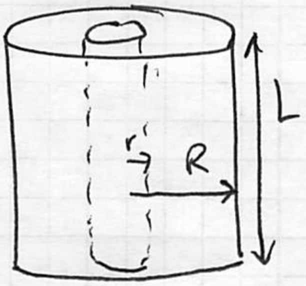
* Clicker Question! Assume this flow fills all space, what can you say about the fluid flux through the x-y plane (all of it)?

Volume Integrals

$$\int_V v\,d\tau \quad \left(\begin{array}{c}\text{not often}\\ \int_V \vec{v}\,d\tau\end{array}\right)$$

Example: Determine the total mass of a rod w/ outer radius R, inner radius, r, length L and mass density $\rho_0 \phi/\phi_0$, determine the total mass of the rod.

* Clicker Question! What coordinate system makes sense?



$\rho = \rho_0 \phi / \phi_0$

use cylindrical coordinates b/c the limits of the integrals you will perform are easily known *

* One of the big things to learn in this class will be which system of coordinates to use when solving different problems. At different times those choices will be informed by,

- symmetries
- geometry of the problem
- independence from certain coordinates
- easily written limits
- functional dependence of one limit on another

As we go through this class, we will discuss those details.

$$M = \int_V \rho \, d\tau \qquad d\tau = s\,ds\,d\phi\,dz \qquad \rho(\phi) = \rho_0 \phi/\phi_0$$

$$M = \int_V \rho_0 \phi/\phi_0 \, s\,ds\,d\phi\,dz = \frac{\rho_0}{\phi_0} \int_r^R s\,ds \int_0^{2\pi} \phi\,d\phi \int_0^L dz$$

$$M = \frac{\rho_0}{\phi_0} \left(\frac{R^2 - r^2}{2} \right) \left(\frac{1}{2} (2\pi)^2 \right) (L)$$

$$M = \frac{\pi^2 \rho_0 L}{\phi_0} (R^2 - r^2)$$

Question:

Do the units make sense?

In addition to these integrals, we will have to dust off our knowledge of different kinds of derivatives. In 481, we will also extend this understanding to more visual and conceptual descriptions of these derivatives as they will help us think through different kinds of problems & models.

Derivative of scalar function - single variable

$$f(x) \rightarrow df = \left(\frac{df}{dx}\right)dx$$

tells us how much f changes in a little step dx.

You know alot about this kind of derivative, e.g., $\frac{df}{dx} \rightarrow$ slope of $f(x)$ at any pt.

What about a function of three variables?

Derivative of scalar function - three variables

$T(x,y,z)$? If we take an arbitrary step in 3-space, $d\vec{l} = \langle dx, dy, dz\rangle$ then,

$$dT = \left(\frac{\partial T}{\partial x}\right)dx + \left(\frac{\partial T}{\partial y}\right)dy + \left(\frac{\partial T}{\partial z}\right)dz$$

How much does T change in a little $d\vec{l}$ step?

This change can be constructed as a dot product

$$\left\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right\rangle \cdot \langle dx, dy, dz\rangle = \nabla T \cdot d\vec{l}$$

∇ is the "del operator"; its not a vector but rather a vector operator
— it does things to scalar and vector functions.

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

- it can act on scalar & vector functions much like $\frac{d}{dx}$ & $\frac{d}{dt}$ do.
- much like $\frac{df}{dx}$ gives the slope of the function f, ∇T ~~gives~~ points in the direction of maximum increase of T.
- As a vector operator, it can act on vectors in the ways you have seen before:

"dot" $\nabla \cdot \vec{V}$ (divergence)

"cross" $\nabla \times \vec{V}$ (curl)

Divergence of a vector field

$$\nabla \cdot \vec{V} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle V_x, V_y, V_z \rangle$$

$$\nabla \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

The divergence is a measure of how much the vector diverges from the point in question.

$f(x,y,z) = \nabla\cdot\vec{V}$ is a scalar function $\left(\begin{smallmatrix}\text{scalar}\\ \text{field}\end{smallmatrix}\right)$ that gives this measure at every point.

$f(x_0,y_0,z_0) = \nabla\cdot\vec{V}_{(x_0,y_0,z_0)}$ is a number, the divergence at the point (x_0,y_0,z_0)

<u>Visualizing divergence</u>: think of a flow field with sawdust sprinkled on top, does the sawdust collect or expand out? probably divergent!

* Clicker Question: Which of the following have $\nabla\cdot\vec{V}=0$?

Some help: Can you think of a plausible vector field for I & II? $\vec{V}(x,y,z)$?

I: II: Both point in $-\hat{x}$, but one depends on x & the other on y ...

Curl of a vector field

$$\nabla\times\vec{V} = \det\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ V_x & V_y & V_z\end{vmatrix}$$

The curl is a measure of how much the vector "curls around" the point in question.

$\vec{f}(x,y,z) = \nabla\times\vec{V}$ is a vector function $\left(\begin{smallmatrix}\text{vector}\\ \text{field}\end{smallmatrix}\right)$ that gives this measure at every pt.

$\vec{f}(x_0, y_0, z_0) = \nabla \times \vec{V}_{(x_0, y_0, z_0)}$ is a vector that gives the measure at (x_0, y_0, z_0)

<u>Visualizing</u> <u>curl</u>: think about dropping a paddle wheel $\ast$ into a flow. Does the paddle wheel turn? probably curl.

☆ Clicker Question: Which field has zero curl?

<u>What about $\nabla \cdot \vec{v}$ & $\nabla \times \vec{v}$ in other coordinate sys.?</u>

Look it up! Did I mention the front fly leaf?

<u>Example</u>: pt. charge at origin

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \qquad \text{use spherical coordinates}$$

$$\nabla \cdot \vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) + \ldots \qquad \underline{\text{only } E_r}$$

$$= \frac{q}{\ldots}\ \frac{1}{\ldots}\ \frac{d}{\ldots}\left(r^2 \frac{1}{r^2}\right) = \ ? = 0\ ??$$

Skip if need to

it seems to diverge from the origin...

We will come back to this and add more to the puzzle.

Finally, we will make use of several integral theorems. Some of these define the Maxwell's equations, others are incredibly use in recasting a problem so that it can be solved.

Gradient Theorem

$$dT = \nabla T \cdot d\vec{l}$$ integrate this along a path

$$\int_{\vec{a}\,P}^{\vec{b}} \nabla T \cdot d\vec{l} = T(\vec{b}) - T(\vec{a})$$

This is just a line integral of a vector field, where the field is defined as ∇T.

$$\int_{\vec{a}}^{\vec{b}} \vec{V} \cdot d\vec{l} \quad \text{where} \quad \vec{V} = \nabla T$$

Corrollary 1: B/c the solution to this is just $T(\vec{b}) - T(\vec{a})$ [much like one-D integral $(f(b) - f(a))$] the integral is path indepent.

Corrollary 2: $\oint_P \nabla T \cdot d\vec{l} = 0$ b/c start & end are the same.

Divergence theorem (Also known as Gauss' & Green's thm.)

$$\int_V (\nabla \cdot \vec{V})\, d\tau = \oint_S \vec{V} \cdot d\vec{A}$$

lhs: integral of a derivative (over a volume)
rhs: integral of the value @ the boundary (at surface)

For our purposes, this thereom will be one of the most used early on,

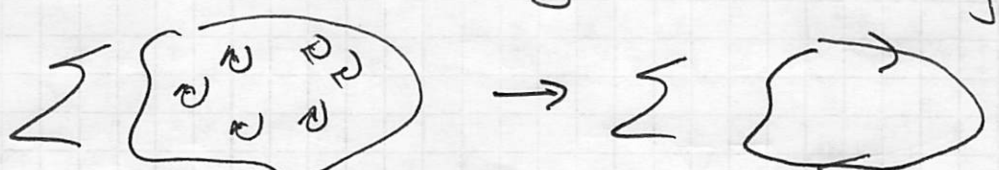
$$\int_V (\nabla \cdot \vec{E})\, d\tau = \oint_S \vec{E} \cdot d\vec{A}$$

Stoke's theorem

$$\int_S (\nabla \times \vec{v}) \cdot d\vec{A} = \oint_C \vec{v} \cdot d\vec{l}$$

lhs: integral of a derivative (flux of the curl over a surface)

rhs: value at the boundary (net circulation of the boundary)



☆ Clicker Question: if $\vec{v} = \nabla T$, $\oint_C \vec{v} \cdot d\vec{l} = ?$

Question: how do you make sense of this?