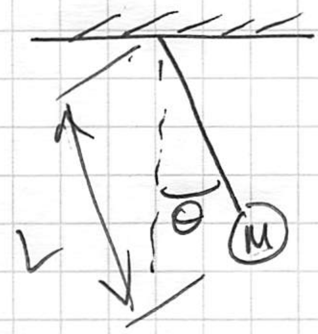
The Harmonic Oscillator gets a bad rap.

Consider a Pendulum with mass, M, a length, L



Through a variety of analyses we can show that

$$mL\ddot{\theta} = -gm\sin\theta$$

or

$$\ddot{\theta} = -\frac{g}{L}\sin\theta$$

we often very quickly limit ourselves to small oscillations (i.e., θ small) So that $\sin(\theta) \simeq \theta$ and thus,

$$\ddot{\theta} \simeq -g/_L\,\theta \quad \text{with } \omega^2 = g/_L$$

then

$$\theta(t) = A\cos(\omega t) + B\sin(\omega t)$$

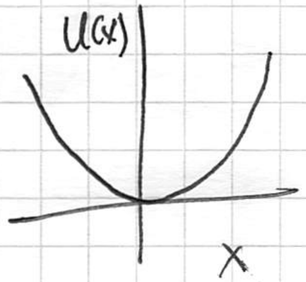
This approximation gets a bad rap b/c it's widely useful (in "limited" contexts):

- → circuits w/ inductor and cap: $\ddot{Q} = -\frac{1}{CL}Q$
- → spring mass: $\ddot{x} = -k/_m\,x$
- → water in U tube: $\ddot{y} = -\frac{2g\rho A}{M}y$
- → jump rope: $\ddot{u} = -\frac{T}{\lambda}\left(\frac{n\pi}{d}\right)^2 u$

etc!

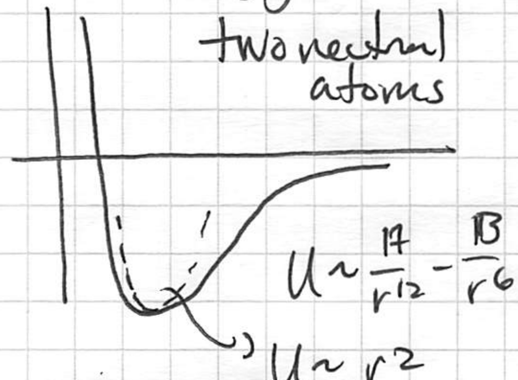
Anything with linear restoring force

② But also, nature tends to be energy minimizing.



$U = \frac{1}{2}kx^2$

Harmonic Oscillator $\Rightarrow$



Gives great help near energy minima. $\Longleftarrow$ Boom! SHO near energy minimum

What if we want more though?

Let's go back to the pendulum,

$\ddot{x} = -\sin x$ where I've absorbed ω^2 in time (or set to 1)

What do we do?

find $x(t)$? But $x(t)$ depends very much on x_0 & v_0. so could get many trajectories ($x(t)$'s)

Enter Dynamical Systems!

→ don't solve for specifics

→ characterize lots of solutions @ once

→ look for qualitatively different behavior

Let's go back to the approximate form,

$\ddot{x} = -x$

We can begin to characterize a whole bunch of solutions by considering a phase space.

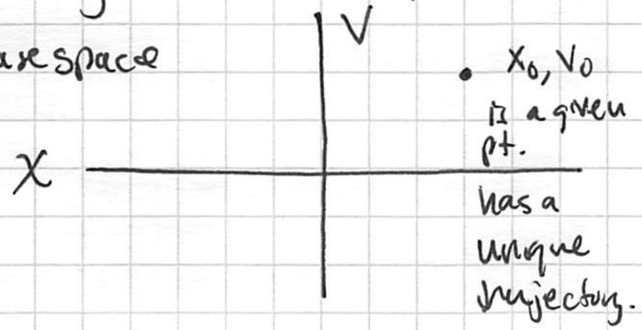
③ Phase Space

→ a space in which all possible states of a system can be shown.

⇒ each state is a unique pt in the space.

for a second order differential equation, we only need two points x & v (think: x_0, v_0 known means $x(t)$ is a unique soln)

phasespace



Change differential equation b/c we need one for x & one for v.

One 2^{nd} order ODE ⟺ Two Coupled 1^{st} order ODEs.

$$\ddot{x} = -x \quad \Longleftrightarrow \quad \begin{aligned} \dot{v} &= -x \\ \dot{x} &= v \end{aligned}$$

Map this to phase space

$$\langle \dot{x}, \dot{v} \rangle = \langle v, -x \rangle$$

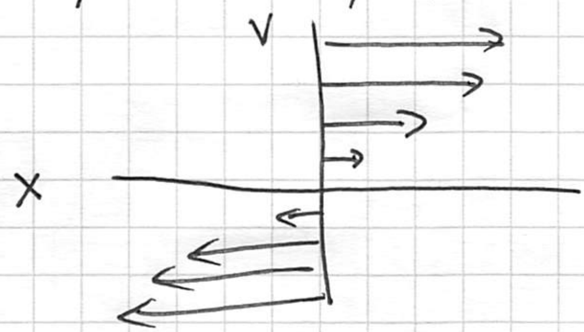
how x & v change ~ where you are in space

Consider: $x = 0$ line.

$$\langle \dot{x}, \dot{v} \rangle = \langle v, 0 \rangle$$

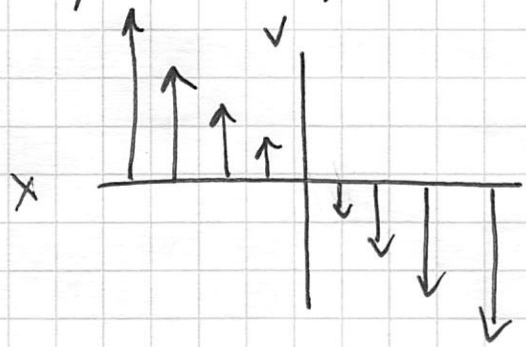
→ v does not change

→ x changes as v changes (same sign)



4 Consider: $v=0$ line

$\langle \dot{x}, \dot{v} \rangle = \langle 0, -x \rangle$ → x does not change
→ v changes as x does (negative sign)



$\Longrightarrow$

Put it together and connect "the dots"...

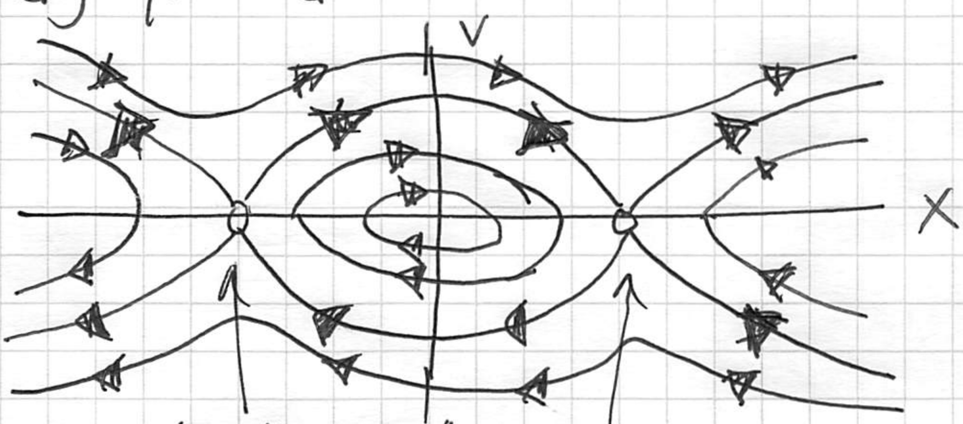
Cool! ✓ All solutions are oscillatory
✓ Total energy is conserved
✓ Each loop characterizes all initial conditions with given energy

But! At some point this breaks down, energy is so large small oscillations is no good!

and then we have $\ddot{x} = -\sin x$ 2nd order

or, $\dot{v} = -\sin x$
$\dot{x} = v$ 2 1st order coupled ODEs

At low energy structure looks similar, but high energy quite different.



"saddle points"

⑤

We get new families of solutions!

(1) periodic, but not sinusoidal

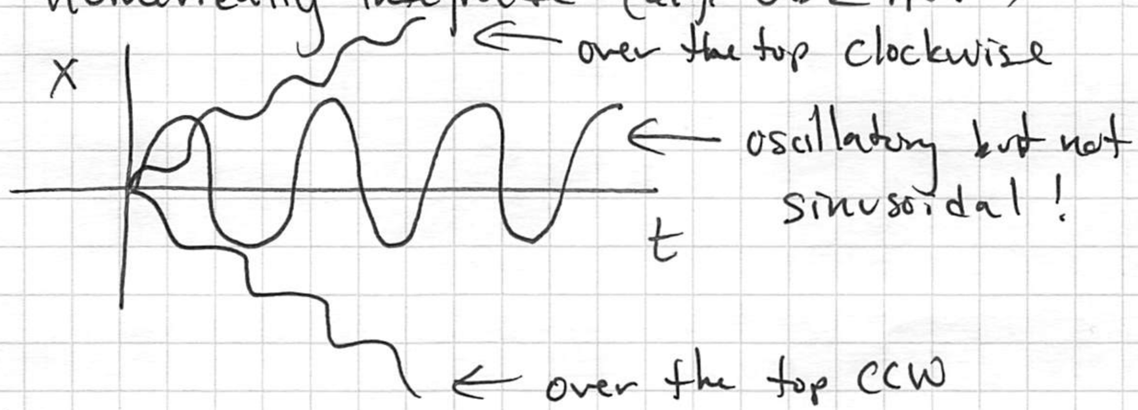
$\Rightarrow$ for very small x & v $\rightarrow$ near sinusoidal

(2) clockwise rotations over the top (v always > 0)

(3) counterclockwise rotations over the top (v always < 0)

Ok but what about specific trajectories?

numerically integrate (e.g. ODEINT)



What about Damped Motion?

$\ddot{x} = -b\dot{x} - \sin x$ approx $\ddot{x} = -b\dot{x} - x$

same approach,

Exact Phase Space

$\dot{v} = -bv - \sin x$

$\dot{x} = v$

Approx

$\dot{v} = -bv - x$

$\dot{x} = v$

new phenomenon an attractor!