Until now, we dealt with motionless (static) source charges. We've found that such static charges will generate electric fields.

Now, we allow those charges to move (with constant speed). Moving charges generate a new kind of field – the magnetic field, $\vec{B}$, which acts in different ways that the electric field, $\vec{E}$, but as we will see in 482 is strongly connected to the electric field.

The magnetic fields that we will concern ourselves with will not change with time, $\vec{B}(\vec{r},t) = \vec{B}(\vec{r})$. We call this field of study, magnetostatics.

<u>A little History</u>

Until the 1800's magnetism was thought of as a distinct force of nature that was not related to electricity.

The observation that drove the development of Magnetic theory was the attraction of lodestones to eachother, to certain materials, and not to others. These observations suggested that the force was <u>not</u> electric as anything with charge will be attracted or repelled to anything else with charge (not material dependent).

Observing a magnetic field is as simple as having a small piece of magnetizable material that can rotate. The material will tend to align with the local magnetic field.

→ This lead to the development of the Compass in the 1500s. Mercator found in 1568 that compasses point to some terrestrial source. (Same guy responsible for the world maps we all use)

→ In 1600, Gilbert proposed the Earth is a big lodestone. 90 yrs. before Newton derives gravitation.

→ Magnetism was thought of as distinct from electricity until the 1820s when Oersted observed that currents produced magnetic effects on compasses.

Lorentz Force Law (Model of forces on charges)

In the prescence of magnetic fields, $\vec{B}$ we observe charges experience forces that are well modeled by the following:

$$\vec{F}_{mag} = q\vec{v} \times \vec{B}$$

this force is derived purely from experiments, its a fundamental model like $\vec{F} = q\vec{E}$.

$\vec{B}$ is the magnetic field with units $1\,\text{Tesla} = 1\,N/(C\,m/s)$

The full form of the Lorentz Force, which describes the force on a charge, q, in the presence of $\vec{E}$ & $\vec{B}$ is,

$$\vec{F} = q\vec{E} + \underbrace{q\vec{v}\times\vec{B}}$$

CQs: charge in $\vec{E}$ & $\vec{B}$

* recall how the right hand rule tells you about the direction of the force!

<u>What is the magnetic field, $\vec{B}$</u>

Magnetic fields are both very different and also very similar to electric fields. This stems from the fact that they are, indeed, one in the same, as we will see in 482.

In static situations,

$\nabla\cdot\vec{E} = \rho/\epsilon_0 \longleftarrow$ ρ is the source of electric field, which diverges.

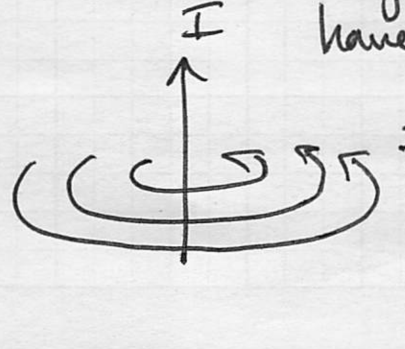
$\nabla\times\vec{B} = \mu_0\vec{J} \longleftarrow$ $\vec{J}$ is the source of $\vec{B}$, which curls
(current)

$\mu_0 = 4\pi\cdot 10^{-7}\ N/A^2$ $\quad$ $1A = 1\text{amp} = 1C/s$.

↑ permeability of free space (value is exact!)

[in fact it defines the amp and thus the coulomb!]

$\nabla\cdot\vec{B} = 0$ $\quad$ magnetic fields have no divergence $\Rightarrow$ there are no magnetic monopoles



$\vec{B}$ curls around sources.

What do magnetic fields do the charges?

Because $\vec{F} = q\vec{v} \times \vec{B}$, magnetic fields do no work; they cannot increase/decrease the kinetic energy of charges $\Rightarrow$ they act to redirect charges.

Proof:

$\vec{F} = q\vec{v} \times \vec{B}$ $\quad\quad dW = \vec{F} \cdot d\vec{l}$ with $d\vec{l} = \vec{v}dt$

Quick

$$dW = q\underbrace{(\vec{v} \times \vec{B})}_{\perp \text{ to } \vec{v} \,\&\, \vec{B}} \cdot (\vec{v}dt) = 0$$

thus zero dot product

Longer

$$dW = q(\vec{v} \times \vec{B}) \cdot (\vec{v} dt) = q dt\, \vec{v} \cdot (\vec{v} \times \vec{B})$$

$$= q dt\, \underbrace{\vec{B} \cdot (\vec{v} \times \vec{v})}_{\text{vector identity.}} = 0 \quad \text{b/c} \quad \vec{a} \times \vec{a} = 0 \text{ for any vector } \vec{a}.$$

Because magnetic fields bend trajectories of charged particles, they are particularly good for "containing" charged particles in a given space.

Consider a "basic" cyclotron, which is just a region of constant (magnitude & direction) magnetic field. Suppose $\vec{B} = B_0\hat{z}$ thus $F_z = 0$ so any motion in the z-direction will be constant velocity; it just "drifts". For the sake of this example the charge starts with $v_z = 0$.

$\vec{F} = \frac{d\vec{p}}{dt} = q\vec{v}\times\vec{B}$ with $\vec{p} = |\vec{p}|\hat{p}$,

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}\left(|\vec{p}|\,\hat{p}\right) = \hat{p}\frac{d|\vec{p}|}{dt} + |\vec{p}|\frac{d\hat{p}}{dt}$$

magnetic fields do no work so $|\vec{p}|$ will not change. thus $\rightarrow$ $\frac{d\vec{p}}{dt} = |\vec{p}|\frac{d\hat{p}}{dt}$

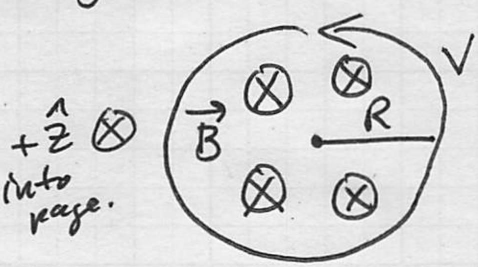
only the direction will change, the speed remains constant $\Rightarrow$ uniform circular motion.

$\vec{v}\perp\vec{B}$. b/c $\vec{v}$ is in the plane, so that

$$\vec{F} = \frac{d\vec{p}}{dt} \Longrightarrow qvB = ma_c = mv^2/R$$

centripetal accel.

Viewing from below, we see,



the radius of the circular trajectory can be derived,

$$R = \frac{mv}{qB} = \frac{|\vec{p}|}{qB}$$

This motion around the circle is steady (i.e., we can measure a simple period of the orbit).

$$2\pi R = vT \Rightarrow T = \frac{2\pi R}{v}$$

Thus, for a given charge experiencing a known magnetic field we can derive the frequency of the orbit,

$$f = \frac{1}{T} = \frac{v}{2\pi R}$$

$$f_{cyclotron} = \frac{qBR/m}{2\pi R} = \frac{qB}{2\pi m}$$

This frequency depends only on the charge and the magnetic field! It's independent of the speed and radius of the orbit!

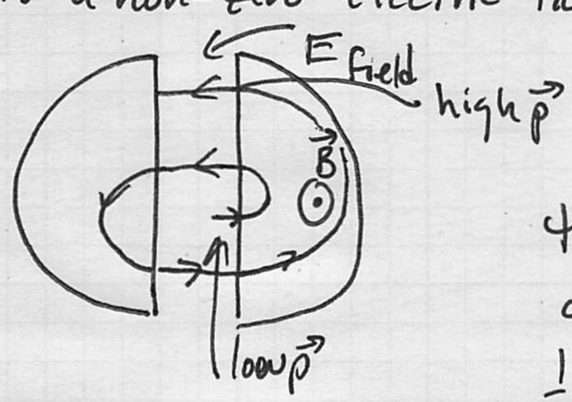
same time to travel the circles

CQ: if $v_z \neq 0$?

When $v_z \neq 0$, the charge drifts along z at constant speed $\Rightarrow$ helical trajectory

"Real" Cyclotrons

Because $\vec{B}$ can't accelerate charges, we need $\vec{E}$ to do that. A real cyclotron consists of two half circles where there is a magnetic field. The gap between the half circles is where there is a non-zero electric field to accelerate charges.



inject low $\vec{p}$ particles and get high $\vec{p}$ out.

The Electric field cycles direction with a frequency

$$\frac{1}{2} f_{cyclotron} = \frac{qB}{2\pi m} \cdot \frac{1}{2}$$

the charges accelerate every time they cross the gap and the magnetic field keeps them contained and directs them back to be accelerated.

- Works up to relativistic speeds, but might have to increase $\vec{B}$.

[Fermilab gets charges to 10^{12} eV (TeV) using $R \approx 2$ km]

Example: Mass Spectrometer

One important application of the Lorentz force is its use in identifying the composition of some material sample. Because different atoms have different masses, the clever use of the Lorentz force can allow us to detect those different components.

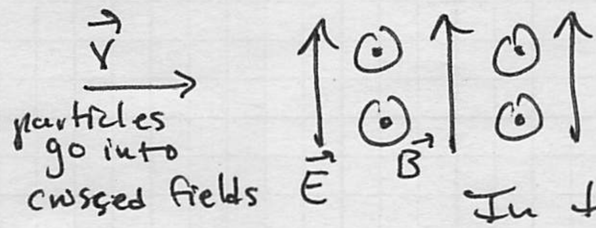
There are two stages to a mass spectrometer.

Stage 1: velocity selector

Stage 2: basic cyclotron + detector.

Stage 1: Velocity Selection

CQ: speed of particles



$\vec{B}$ out of page

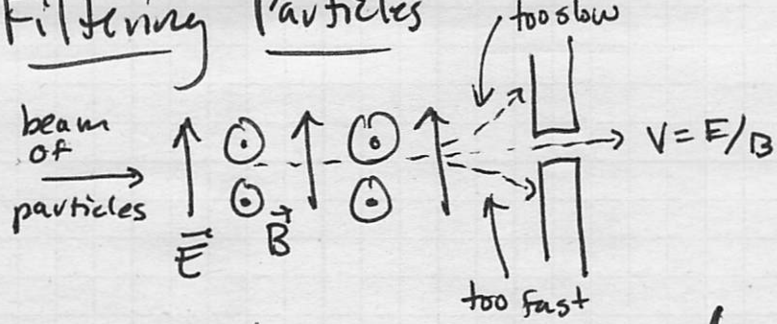
$\vec{E}$ upward

In this situation, a positive charge feels an electric force upward and a magnetic force downward.

$$\vec{F} = q\vec{E} + q\vec{v}\times\vec{B} \Leftarrow$$

But a particle moving with a specific $v = E/B$ will experience no net force (up or down), so it will move through the space with both fields at constant speed; all other velocities will result in some curved trajectories as the magnetic + electric forces will not be of equal size. By placing a physical barrier at the end of the region we can "filter" particles.

Filtering Particles

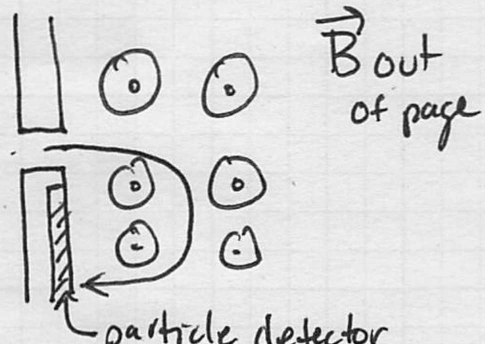


So particles that arrive on this side all have a known speed $V = E/B$ (even though we don't know their masses)

CQ: filter Question

Stage 2: Basic Cyclotron + Detector

After particles are filtered, the all move with the same speed. We can exploit their curved motion in magnetic fields to separate them by mass and detect them.



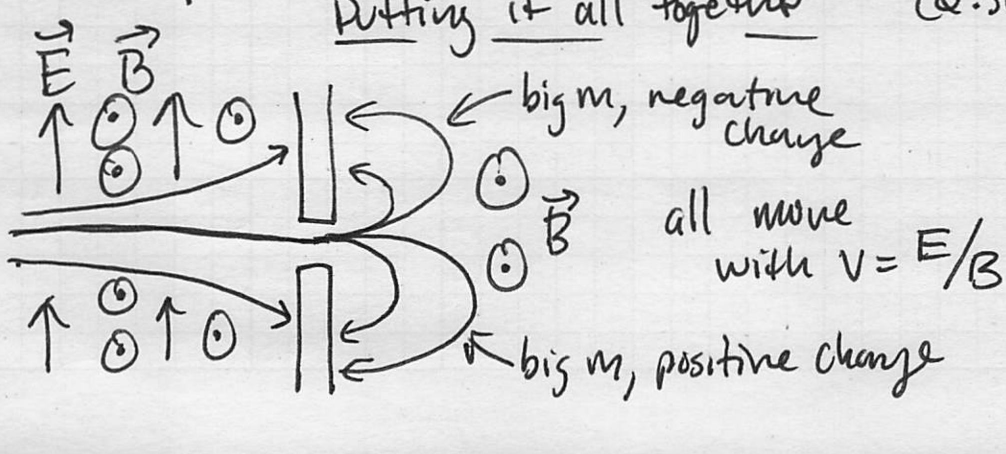
particles will move in a semicircular arc. If we know q, which is easy to determine from just E field then,

$$R = \frac{|\vec{p}|}{qB}$$

So by measuring where on the detector you observe a signal, with a known B, you can determine, m. Thus, characterize the sample.

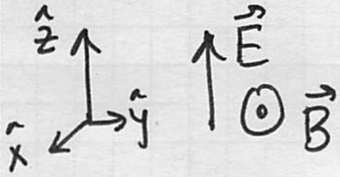
Putting it all together

CQ: signs of charge



Trajectories in the velocity selector

Let's release a charge (q, m) from rest in a space where the electric and magnetic fields are given.



$$\vec{E} = E_0 \hat{z} \qquad \vec{F} = q\vec{E} + q\vec{v}\times\vec{B} = m\vec{a}$$
$$\vec{B} = B_0 \hat{x}$$

All these forces will be in the y-z plane so $F_x = 0$ and thus $v_x = 0$ b/c start from rest.

We expect velocities in the y-z plane.

$$\vec{v} = \langle 0, v_y, v_z \rangle$$

$$m\vec{a} = m\langle 0, \dot{v}_y, \dot{v}_z \rangle = q\langle 0, 0, E_0 \rangle + q\vec{v}\times\vec{B}$$

$$\vec{v}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & v_y & v_z \\ B_0 & 0 & 0 \end{vmatrix} = \langle 0, +v_z B_0, -v_y B_0 \rangle$$

$$m\vec{a} = m\langle 0, \dot{v}_y, \dot{v}_z \rangle = \langle 0, q v_z B_0, qE_0 - q v_y B_0 \rangle$$

so,

$$m\dot{v}_y = q v_z B_0 \qquad m\dot{v}_z = qE_0 - q v_y B_0$$

CQ: how to solve?

How might we find solutions to these coupled differential equations? Attempt to eliminate the coupling?

$$m\ddot{v}_y = qB_0 \dot{v}_z \quad \text{plug this into equation for } \dot{v}_z$$

$$\dot{v}_z = \frac{m\ddot{v}_y}{qB_0} \Rightarrow m\left(\frac{m\ddot{v}_y}{qB_0}\right) = qE_0 - q v_y B_0$$

So we can uncouple the differential equation for v_y,

$$\ddot{v}_y = \frac{q^2 B_0}{m^2}\left(E - Bv_y\right)$$

this differential equation is of the form,

$$\ddot{x} = a - bx \quad \text{with} \quad a = \frac{q^2}{m^2} B_0 E_0 \quad \text{and} \quad b = \frac{q^2 B_0^2}{m^2}$$

The general solution to this differential equation is,

$$x = \tilde{C}_1 \sin(\sqrt{b}\,t) + \tilde{C}_2 \cos(\sqrt{b}\,t) + a/b = v_y$$

But $x = dy/dt = v_y$ so,

$$y = C_1 \cos(\sqrt{b}\,t) + C_2 \sin(\sqrt{b}\,t) + \frac{a}{b} t + C_3$$

where absorbed factors of $\sqrt{b}$ and signs into C_1 & C_2.

and $\sqrt{b} = \frac{qB_0}{m}$ $\quad a/b = E_0/B_0$

So without putting in the initial conditions,

$$y(t) = C_1 \cos\left(\frac{qB_0}{m} t\right) + C_2 \sin\left(\frac{qB_0}{m} t\right) + E_0/B_0\, t + C_3$$

<u>What about v_z & $z(t)$?</u>

$$v_z = \frac{m}{qB_0} \dot{v}_y = \frac{m}{qB_0}\left(-\frac{q^2 B_0^2}{m^2}\left(C_1 \cos(\sqrt{b}\,t) + C_2 \sin(\sqrt{b}\,t)\right)\right)$$

$$v_z = -\frac{qB_0}{m}\left(C_1 \cos(\sqrt{b}\,t) + C_2 \sin(\sqrt{b}\,t)\right)$$

$$z(t) = \int v_z\, dt = -C_1 \sin(\sqrt{b}\,t) + C_2 \cos(\sqrt{b}\,t) + C_4$$

So without putting in initial conditions,

$$z(t) = -C_1 \sin\left(\frac{qB_0}{m} t\right) + C_2 \cos\left(\frac{qB_0}{m} t\right) + C_4$$