- So far our job in electrostatics has been to find $\vec{E}$ given the charges that generate it.

- Why do we care about $\vec{E}$? Because $\vec{F} = q\vec{E}$ gives us the force on other charges and helps describe the motion and ultimately influence/control the motion of charged particles
  $\Rightarrow$ lots of important applications!

We've developed two methods so far,

(1) If we know $\rho(\vec{r})$, we can calculate $\vec{E}$ directly,

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\, d\tau}{\mathscr{r}^2}\hat{\mathscr{r}}$$

this seems like a straightforward calculation, but in practice it's hard for most configurations.

It becomes super problematic when ρ changes in some hard to establish way $\rightarrow$ like in a conductor!

(2) If we know V, we can calculate $\vec{E} \Rightarrow \vec{E} = -\nabla V$

But V comes from ρ,

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\, d\tau}{\mathscr{r}}$$

this is arguably easier than computing $\vec{E}$, but can still be quite difficult

(And it is still problematic when ρ changes in some way)

A Third method that we have alluded to, but haven't yet worked with is Poisson's Equation for V!

$$\nabla^2 V = -\rho/\epsilon_0$$

- This differential equation can be tough to solve, but in regions where there's no charge, it becomes a bit easier. $\rho = 0$, $\nabla^2 V = 0$ Laplace's Equation

[Clicker Q: $\rho = 0$]

- Now it might look inoccuous, but this PDE is one of the more ubiquitous ones in physics. It shows up in:
  - Heat Flow
  - Hydrodynamics
  - Diffusion

- The way to solve it is to know its value at the boundary (or it's derivative) and to use those "boundary conditions" to set coefficients on a general solution (much like ODEs).

- Once you find $V \rightarrow \vec{E} = -\nabla V$ in that region

Solving Laplace's Equation

$$\nabla^2 V = 0 \quad \text{in Cartesian is,}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

- Typically, we will guess an ansatz (possible solution), which will generate a general solution with unknown coefficients.

- these coefficients will be set by the boundary conditions (matching them)

- By the uniqueness thm, this will be our solution.

These methods to develop a solution to Laplace's equation are applicable in other areas of physics
Quantum Mechanics, plasma physics, travelling waves

Before we solve an example problem
Let's talk about:

Properties of solutions to $\nabla^2 V = 0$

(These are all provable, but we will just use them)

① V has no local maxima or minima anywhere but on the boundaries

② V is smooth and continuous everywhere.

③ V at a location is equal to the average V over any surrounding sphere

$$V(\vec{r}) = \frac{1}{4\pi R^2} \underset{\substack{\text{sphere of} \\ \text{radius } R \\ \text{centered on } \vec{r}}}{\oiint} V\, dA$$

④ $V(\vec{r})$ is unique: If $\nabla^2 V = 0$ and you know (the boundary conditions, either V or $\partial V/\partial \hat{n}$ on boundary) then your solution is unique

$\hat{n}$ normal

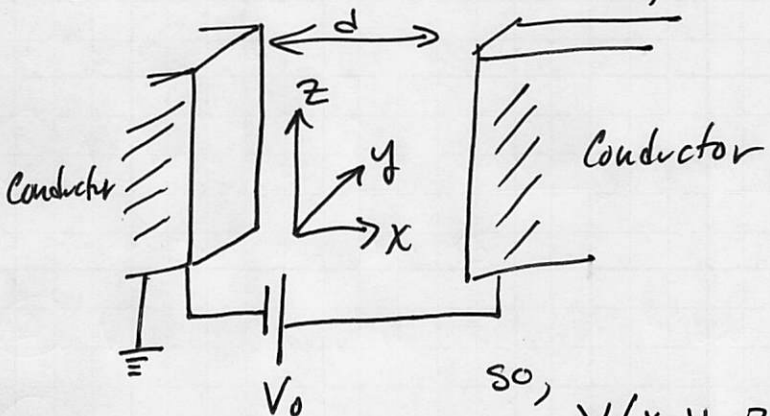
We will prove this b/c it's so powerful! $\Leftarrow$

Clicker Question: $\rho = 0$; $V = 0$ @ boundary

<u>Example</u>: $\nabla^2 V = 0$ in one dimension

Consider a pair of conductors with flat rectangular faces. the one on the left is grounded and the one on the right is at a potential V_0.

It looks like this,



This is a 1D problem as V cannot depend on y or z given the boundary conditions.

so,
$$V(x,y,z) = V(x)$$

Because $V(x,y,z) = V(x)$,

$$\nabla^2 V = 0 \implies \frac{d^2 V(x)}{dx^2} = 0$$

Our boundary conditions are $V(0) = 0 \quad V(d) = V_0$

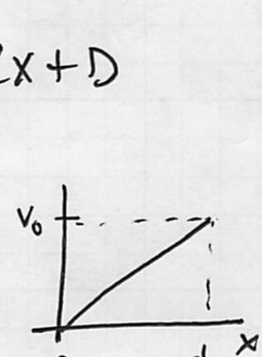
★ <u>Activity</u>: Find the solution.

$$\frac{d}{dx}\left(\frac{dV}{dx}\right) = 0 \Rightarrow \frac{dV}{dx} = C \leftarrow \text{constant}$$

so general solution $V(x) = Cx + D$

But with $V(0) = 0 \Rightarrow D = 0$

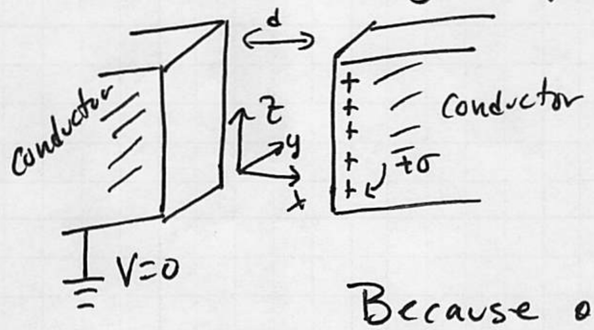
$V(d) = V_0 \Rightarrow C = V_0/d$



$V(x) = V_0 x/d$ (a capacitor)

Note: V is smooth, simple, and boring.
It has a maxima <u>only</u> at the edges.
average value at the middle

<u>Example</u>: Same problem with different boundary cond.
(specify charge instead of potential)



We have dumped charge on the conductor on the right. ⇒ we measure $+\sigma$ on that wall.

Because of specifying $+\sigma$ we know $\vec{E}$ just outside the conductor at $x=d$.

$$\Delta E_\perp = \sigma/\epsilon_0 \quad \text{b/c} \quad E=0 \text{ in the metal}$$

$$E_{\text{just outside}} = \sigma/\epsilon_0$$

or more precisely,

$$\vec{E}(x=d) = -\frac{\sigma}{\epsilon_0}\hat{x}$$

B/c $\vec{E} = -\nabla V \Rightarrow \left.\frac{dV(x)}{dx}\right|_{x=d} = +\frac{\sigma}{\epsilon_0}$ is our new Boundary Condition

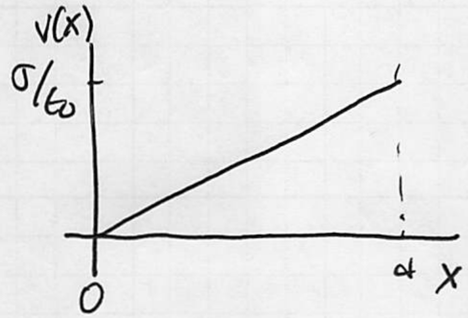
So, $\nabla^2 V = 0 \Rightarrow \frac{d^2V}{dx^2} = 0$ b/c $V(x,y,z) = V(x)$ as before

So,

$V(x) = Cx + D$ as before with $V(0) = 0$
$D = 0$

$V(x) = Cx \qquad \frac{dV}{dx} = C$ so $C = \frac{\sigma}{\epsilon_0}$

And $V(x) = +\frac{\sigma}{\epsilon_0}x$



This has the same functional form as our previous solution, but it depends on the boundary conditions for setting constants.

Reminders of General Properties

① V has no local min or max (except at boundary)
② V is smooth & continuous everywhere
③ $V(\vec{r}) = \frac{1}{4\pi R^2} \oint V \, dA$ (average value)
④ $V(\vec{r})$ is unique if $\nabla^2 V = 0$ & you have the BC's.

Consequences of these properties

① Because there's no min or max where there's free space, they are no "hills" or "valleys" in the potential.

Analogy: stretch a rubber sheet over some boundary very tight so it doesn't distort. Place a ball and it will fall off (no local min).

Earnshaw's thereom: no charge can be held in stable equilibrium by electrostatic forces alone.

Clicker Question: 

③ Because we can specify $V(\vec{r})$ as the average V of the points around it, we can solve $\nabla^2 V = 0$ computationally.

"Method of relaxation"
- Specify V(r) at boundary
- "Guess" V(r) on grid of points in empty space
- Step through each point taking average of surrounding pts. Repeat!

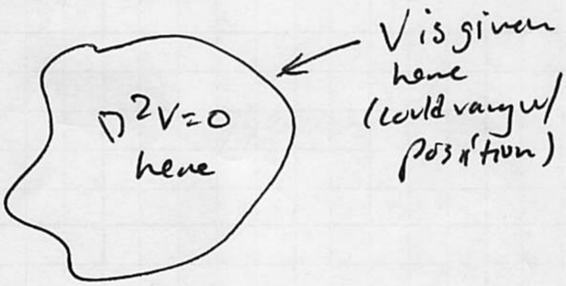
this will produce a numerical approximation of your answer.

Very <u>useful</u> technique and widely applicable

(4) If you can guess the solution that satisfies $\nabla^2 V = 0$ along with the boundary conditions, you've solved your problem! (Also true for Poisson's)

<u>General Property 4</u> <u>almost seems</u> magical!

Let's prove it.



Suppose we have two possible solutions

$$\nabla^2 V_1 = 0 \quad \text{and} \quad V_1(\text{boundary}) = V_2(\text{boundary}) = V_{\text{given}}$$
$$\nabla^2 V_2 = 0$$

So let $W = V_1 - V_2$, then because ∇^2 is a linear operator,

$$\nabla^2 W = \nabla^2 (V_1 - V_2) = \nabla^2 V_1 - \nabla^2 V_2 = 0$$

But, $W(\text{boundary}) = 0$ b/c $V_1(\text{boundary}) = V_2(\text{boundary})$

So, $W = 0$ everywhere and thus $V_1 = V_2$ everywhere

So there's only one possible solution!

→ no local min or max!

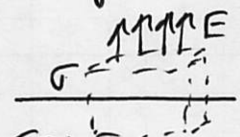
Solving $\nabla^2 V = 0$ relies on knowing your Boundary Conditions to determine your unknown coefficients.

- Either you need V or $\frac{\partial V}{\partial n}$ to find V

(it can be a mix, but careful to not overspecify!)

- Here's the more common Boundary Conditions you will encounter:

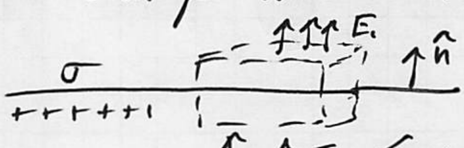
① Near any conductor, we can specify the charge and relate it to $\frac{\partial V}{\partial n}$,



$\vec{E} = \frac{\sigma}{\epsilon_0}\hat{n}$ ← normal

which gives $\frac{\partial V}{\partial n} = -\frac{\sigma}{\epsilon_0}$ $\left(\underset{\text{more general}}{\nabla V \cdot \hat{n} = -\frac{\sigma}{\epsilon_0}} \right)$

② Near any sheet of charges, we can also specify the charge and relate it to $\partial V/\partial n$.



← Note: E_2 might be +/−

Here

$E_1 A - E_2 A = \sigma A/\epsilon_0$ given $\hat{n}$ pointing up.

So, $E_{\substack{\text{normal}\\ \text{above}}} - E_{\substack{\text{normal}\\ \text{below}}} = \sigma/\epsilon_0$ or,

$$\left.\frac{\partial V}{\partial n}\right|_{\text{above}} - \left.\frac{\partial V}{\partial n}\right|_{\text{below}} = -\sigma/\epsilon_0$$

③ $\nabla \times \vec{E} = 0$ implies $E_{\parallel}$ is continuous

$E_{\parallel}$ above $= E_{\parallel}$ below

④ $V_{\text{above}} = V_{\text{below}}$ V is always continuous.