We have solved the Coulomb problem where some source charges produce an electric field at a location $\vec{r}$!

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r}\,')\,d\tau'}{\mathscr{r}^2}\hat{\mathscr{r}}$$

- We can use this description to solve many problems. However, sometimes this description of the problem isn't helpful because the construction of the integral is difficult, the integral is non-analytic, or (as we will see) the problem is posed such that $\rho(\vec{r}\,')$ is unknown.
- It is helpful to develop our <u>theoretical toolbox</u> to be more flexible when solving certain kinds of problems. Our first such new tool is something you've seen before! Gauss' Law

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$$

- Gauss' Law is an experimental "law" of nature
  - we observe that it holds in all contexts
- Can be derived from Coulomb's Law as we will see

<u>Electric Flux</u>, Φ_E

total $\Phi_E = \oint_S \vec{E}\cdot d\vec{A}$ ← this closed surface integral in Gauss' Law is the total electric flux through a volume.

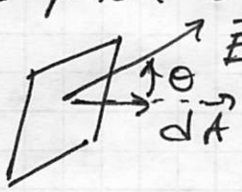
through the closed volume

We can also consider the flux through a surface that is "open"

$$\Phi_E = \int_S \vec{E} \cdot d\vec{A}$$

this is the electric flux through the surface S.

We can think about the kernel of this integral by considering the surface S and little patch of that surface dA.



$d\vec{A}$ is the vector normal ($\perp$) to the surface dA.

thus, $d\Phi_E = \vec{E} \cdot d\vec{A} = E\,dA\cos\theta = E_\perp \cdot \text{Area}$

Flux is proportional to the area and the amount of $\vec{E}$ poking through the surface.

Electric Field Lines

field lines point away from $+q$ and towards $-q$.

these lines represent the electric field.

The density of the lines (in 3D!) represent the magnitude of $\vec{E}$ ($|\vec{E}|$)

Two dimensional electric field pictures can fool you!

so $\dfrac{\#\text{lines}}{\text{area}} \longleftrightarrow |\vec{E}|$

the flux is represented by how many lines "poke" through the area.

☆ Clicker Questions:

- Constant $\vec{E}$ $\oint \vec{E} \cdot d\vec{A}$?
- flux through cylinder.

Gauss' Law almost seems like magic*; we can show that for any surface and any q that,

$$\oint_S \vec{E} \cdot d\vec{A} = q_{enc}/\epsilon_0$$

*Note: it is true that Gauss' Law always holds, but it is not always useful!

Consider some nastly shape with an enclosed charge,

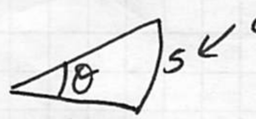
We can prove that

$$\oint_S \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0}$$

for this arbitrary shape.

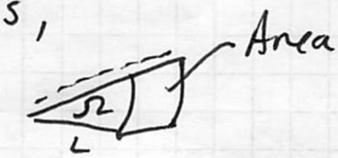
Let's first unpack the kernel of the integral, by constructing dA,

you might remember,



$\theta \equiv \frac{s}{L}$ definition of an angle

the "solid angle" is the 2D generalization of this,

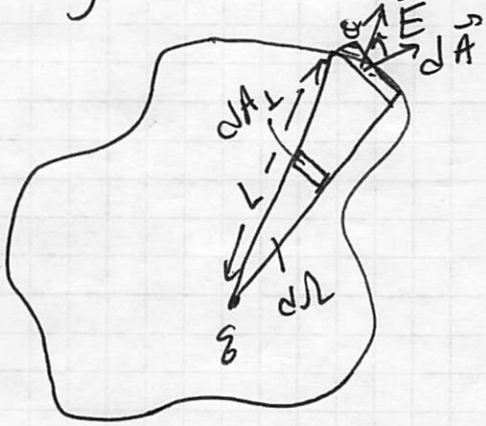


$$\Omega \equiv \frac{Area}{L^2}$$

For very small areas, the surface is basically flat, $dA_\perp$

$$d\Omega = \frac{dA_\perp}{L^2}$$

Going back to our arbitrary shape,



The little contribution to the flux from this patch is,

$$\vec{E}\cdot d\vec{A} = \frac{q}{4\pi\epsilon_0}\frac{\cos\theta\, dA}{L^2}$$

But notice that!

$$d\Omega = \frac{dA_\perp}{L^2} = \frac{dA\cos\theta}{L^2}$$

that is the thing that varies is the solid angle

so,

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{q}{4\pi\epsilon_0}\int d\Omega$$

the solid angle integral over <u>any</u> closed surface is 4π

$$= \frac{q}{4\pi\epsilon_0}(4\pi) = q/\epsilon_0 \; !$$

Well this was proved for one charge, what about if we add more charges?

<u>Superposition saves us again!</u>

$$\oint_S \vec{E}_{net}\cdot d\vec{A} = \oint_S (\vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \ldots)\cdot d\vec{A}$$

$$= \oint_S \vec{E}_1\cdot d\vec{A} + \oint_S \vec{E}_2\cdot d\vec{A} + \oint_S \vec{E}_3\cdot d\vec{A} + \ldots$$

$$= \frac{q_1}{\epsilon_0} + \frac{q_2}{\epsilon_0} + \frac{q_3}{\epsilon_0} + \ldots = \frac{q_{enc}}{\epsilon_0} \; !$$

Well this is great! the total flux over some closed volume tells us about the <u>net</u> charge in the volume. But where does that come from? Gauss' $\longleftrightarrow$ Coulomb?

How are Gauss' Law and Coulomb's Law related?

Let's go back to some vector calculus we seen before → the divergence theorem,

$$\int_V \nabla\cdot\vec{F}\, d\tau = \oint_S \vec{F}\cdot d\vec{A} \quad \text{for any vector function } \vec{F}.$$

Meaning: If $\vec{F}$ is any kind of "flow", then
$\vec{F}\cdot d\vec{A}$ is the flux exiting dA
$\nabla\cdot\vec{F}$ is the "divergence". It's the spread from a point or the "creation" of arrows.

* Clicker Questions: Reminders of divergence.

So, $\int_V \nabla\cdot\vec{F}\, d\tau$ is the total spread created by all points in the volume, V.

$\oint_S \vec{F}\cdot d\vec{A}$ is the total outflow through surface S that bounds volume, V.

* What goes out must have originated from sources inside.

So,
$$\oint_S \vec{E}\cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0} = \frac{1}{\epsilon_0}\int_V \rho\, d\tau$$
Here we relate the total charge to the distribution in the volume, V.

But from the divergence theorem,
$$\oint_S \vec{E}\cdot d\vec{A} = \int_V \nabla\cdot\vec{E}\, d\tau$$

so,
$$\int_V \nabla\cdot\vec{E}\, d\tau = \int_V \frac{\rho}{\epsilon_0} d\tau$$
Both integrals occur over the same volume, V.

$$\int_V \left(\nabla\cdot\vec{E} - \frac{1}{\epsilon_0}\rho\right) d\tau = 0 \quad \text{for any } V$$
thus, $\nabla\cdot\vec{E} = \rho/\epsilon_0$

Thus, Gauss' Law is just another way to write Coulomb's Law*

*Note: Actually it is more ubiquitous and it far reaching than that (as we will see later).

We want to use Gauss' Law to solve problems, but before we do that, Let's see if we can describe a point charge appropriately with Coulomb's law,

$$\nabla\cdot\vec{E} = \rho(\vec{r})/\epsilon_0$$

Activity! $\vec{E}(\vec{r}) = \dfrac{q}{4\pi\epsilon_0}\dfrac{\hat{r}}{r^2}$ compute $\nabla\cdot\vec{E}$.

$$\nabla\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = 0!$$

But $\rho(\vec{r}) \neq 0$! What going here??

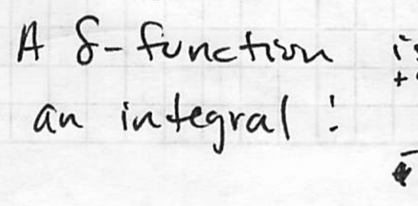
Dirac Delta Functions (to the rescue)

In 1 dimension we define the δ function to be,

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases} \quad \text{such that} \int_{-\text{any}}^{+\text{any}} \delta(x)\,dx = 1$$

(as long as we include 0!)

The delta function is the limit of a sensible function like this,



It's a tall & skinny shape with an area of 1.

as $\epsilon \to 0$ we get a δ function.

A δ-function is actually defined in terms of an integral: $\displaystyle\int_{-\text{any}}^{+\text{any}} f(x)\,\delta(x)\,dx = f(0)$

So a δ-function "picks" out the value of the function at zero. So long as the domain of the integral includes the peak of the delta function it will do this,

$$\int_0^2 f(x)\,\delta(x-1)\,dx = f(1) \quad \text{but,}$$

$$\int_{-2}^0 f(x)\,\delta(x-1)\,dx = 0 \quad \text{b/c the } \delta \text{ function is zero except at } x=1.$$

*Clicker Question: $\int \delta(x) f(x)\,dx$

Example

Consider: $\int_{-\infty}^{\infty} f(x)\,\delta(ax+b)\,dx$ — We deal with this by performing a u-substitution,

Let $u = ax+b$

$du = a\,dx$ so that,

$$\int_{-\infty}^{\infty} f(x)\,\delta(ax+b)\,dx = \int_{-\infty}^{\infty} f\left(\frac{u-b}{a}\right)\delta(u)\,du/a$$

$= \frac{1}{a} f($the x value for which the δ-function argument vanishes, $-\frac{b}{a})$

$= \frac{1}{a} f(-b/a)$

Note: when $a<0$, integration limits are flipped you get $-\frac{1}{a} f\left(\frac{-b}{a}\right)$ so, $\delta(ax) = \frac{1}{|a|}\delta(x)$

Activity: Here's a few δ-functions to integrate. What do you get? Anything you need to be careful about?

$$\int_{-\infty}^{\infty} x e^x \delta(x-1)\,dx = e^1 \qquad \int_{+\infty}^{-\infty} \log(x)\,\delta(x-2)\,dx = -\log(2)$$

$$\int_{-\infty}^{0} x e^x \delta(x-1)\,dx = 0 \qquad \int_{-\infty}^{\infty} (x+1)^2 \delta(4x)\,dx = \frac{1}{4}$$

In 3D we define the δ-functions as,

$$\int_V \delta^3(\vec{r})f(\vec{r})d\tau = f(0)$$ as long a V contains $\vec{r}=0$

So a point charge has:

- no charge density a way from the origin
- infinite density at the origin

But the integral for this distribution is q!

$$q = \int_V \rho(\vec{r})\,d\tau$$ that is q is finite!

So the distribution for a pt. charge is a 3D delta function.

– Charge at the origin: $\rho(\vec{r}) = q\delta^3(\vec{r})$

very "concentrated" charge at origin with finite volume integral.

*Clicker questions: – what about if q is at some other location?
what are the units of $\delta(x)$ & $\delta^3(\vec{r})$?

Ok so we know about δ-functions now and we have a description of a pt. charge as,

$$\rho(\vec{r}) = q\delta^3(\vec{r}) \quad \text{or} \quad \rho(\vec{r}) = q\delta^3(\vec{r}-\vec{r}\,')$$
at origin   at $\vec{r}\,'$

We took this mathematical interlude to understand a conundrum, $\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}$ *CQ: Box w/ divergence

Let's compute $\nabla \cdot \vec{E}$ for the pt. charge.

$$\nabla \cdot \vec{E} = \nabla \cdot \left(\frac{q}{4\pi\epsilon_0} \frac{\hat{r}}{r^2} \right) = \frac{q}{4\pi\epsilon_0} \nabla \cdot \left(\hat{r}/r^2 \right)$$

lets focus on $\nabla \cdot \hat{r}/r^2$ or $\nabla \cdot \vec{r}/r^3$

① Compute using description from front fly leaf,

$$\vec{V} = \frac{\vec{r}}{r^3} = \frac{\hat{r}}{r^2} \qquad \nabla \cdot \vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r)$$

$$v_r = \frac{1}{r^2} \quad \text{so,} \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(0) = 0 \, !$$

ok so we get $\nabla \cdot \vec{E} = 0$

② Let's try being a bit more "careful", let's use the chain rule,

$$\nabla \cdot f\vec{A} = \nabla f \cdot \vec{A} + f\, \nabla \cdot \vec{A} \, ,$$

$$\nabla \cdot \left(\frac{1}{r^3} \vec{r} \right) = \nabla \frac{1}{r^3} \cdot \vec{r} + \frac{1}{r^3} \nabla \cdot \vec{r} = \left(-\frac{3}{r^4} \hat{r} \right) \cdot \vec{r} + \frac{3}{r^3}$$

$$= -\frac{3}{r^3} + \frac{3}{r^3} = 0 \, !$$

So again $\nabla \cdot \vec{E} = 0$. it looks like $\nabla \cdot E = 0$ for a pt. charge!

* Well it is, except at $r=0$ where all these formulas involve r^2/r^2 and that is not well defined!

Let's go back to Gauss' Law,

$$\int_V \nabla \cdot \vec{E} \, d\tau = \oint_S \vec{E} \cdot d\vec{A}$$

Take a very small sphere around the charge

on the right-hand side,

$$\oint_S \vec{E} \cdot d\vec{A} = \oint_S \frac{q}{4\pi\epsilon_0 r^2} \hat{r} \cdot d\vec{A}$$

$$\oint_S \frac{q}{4\pi\epsilon_0 r^2} \underbrace{\hat{r}\cdot d\vec{A}} = \frac{q}{4\pi\epsilon_0 r^2}\int dA = q/\epsilon_0$$

parallel to each other over the whole sphere.

So,

$$\int \nabla\cdot\vec{E}\, d\tau = q/\epsilon_0 \quad \text{no matter what!}$$

it must be that $\nabla\cdot\vec{E}$ is a δ function,

$$\nabla\cdot\vec{E} = \frac{q}{\epsilon_0}\delta^3(\vec{r}) \Rightarrow \rho(\vec{r}) = q\,\delta^3(\vec{r})$$

We also learned a very important relationship,

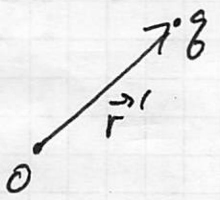
Because,

$$\nabla\cdot\left(\frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}\right) = \frac{q}{\epsilon_0}\delta^3(\vec{r})$$

it must be that,

$$\nabla\cdot\frac{\hat{r}}{r^2} = 4\pi\,\delta^3(\vec{r})$$

To summarize: Point charges can be described by



$$\rho(\vec{r}) = q\,\delta^3(\vec{r}-\vec{r}\,')$$

and

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} \quad \text{as usual}$$

So,

$$\nabla\cdot\vec{E} = \frac{q}{4\pi\epsilon_0}\nabla\cdot\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

$$= \frac{q}{\cancel{4\pi}\epsilon_0}\cancel{4\pi}\,\delta^3(\vec{r}-\vec{r}\,') = \rho(\vec{r})/\epsilon_0$$

as it must be.

We will now apply Gauss' Law to a few different scenarios. Remember,

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$$

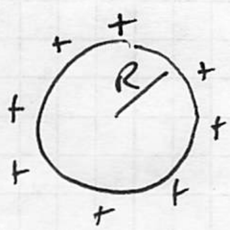
. Gauss' Law is <u>always</u> true, but...

* It is only useful to <u>find E</u> if symmetry let's you "pull out" E.

* Clicker Question: Gaussian surface not a sphere.

<u>Followup question</u>: Can we use Gauss' Law to solve for E? <u>No.</u>

<u>Example</u>: Remember that problem where we had with a spherical shell with uniform σ?



$\vec{E}(\vec{r}) = ?$ From symmetry the electric field points radially

Think: What other direction could it be?

So, $|\vec{E}(\vec{r})|$ only depends on r (not θ or ϕ)

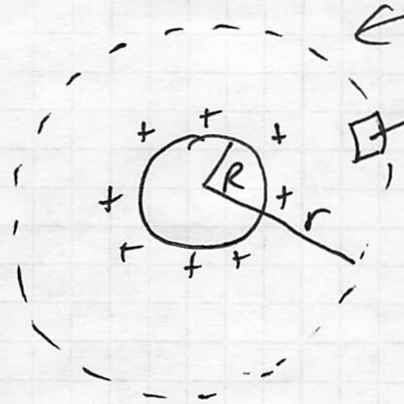
So, $\vec{E}\cdot d\vec{A} = E\,dA$ b/c $\vec{E}$ & $d\vec{A}$ are radial

$\cos\theta \Rightarrow \cos 0 = 1$

how do we make sense of this?

<u>What is my $d\vec{A}$</u>? what surface?

- It is not the sphere; it's (as you may remember) an imaginary sphere
- Gaussian surface is what we call it.



Gaussian Surface (uses symmetry of the problem)

so, $\oint \vec{E} \cdot \vec{dA} = \underbrace{\oint_S E\,dA = E \oint_S dA}$

Gaussian surface, S

↳ Why can we do this?

Important! – Because r is a constant, E is a constant ⇒ everywhere on the sphere.

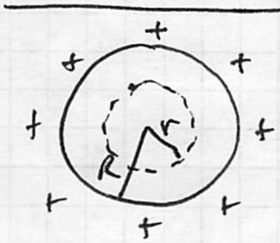
so,

$$\oint \vec{E} \cdot \vec{dA} = E\,4\pi r^2 = \frac{Q_{enc}}{\epsilon_0} \Rightarrow E = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}$$

(with $Q = 4\pi R^2 \sigma$)

so, $\vec{E} = \frac{\sigma R^2}{\epsilon_0 r^2}\hat{r}$ or $\frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r}$ pt charge at the origin

<u>What about inside?</u>



We can use the same arguments here!

$\vec{E}$ is radial and only depends on r.

so, $\oint_S \vec{E} \cdot \vec{dA} = \oint_S E\,dA = E\oint dA = E\,4\pi r^2$

But now, $Q_{enc} = 0$ so, $E = 0$!

★ <u>Clicker Question</u>! Dipole w/ Gaussian Surface.

followup: Can we use Gauss here? No!

What if the sphere had a uniform charge density ρ, rather than just a surface charge, σ?

<u>outside</u> $\Rightarrow$ no difference, you can't tell!
only total Q matters.

<u>inside</u> $\Rightarrow$ $Q_{enc} \neq 0$ anymore, so what is it?

$$Q_{enc} = \int_V \rho \, d\tau = \rho \int_V d\tau = \rho \, \frac{4}{3}\pi r^3$$

so, $$E \cdot 4\pi r^2 = \frac{1}{\varepsilon_0} \rho \, \frac{4}{3}\pi r^3$$

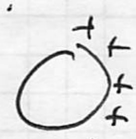
and thus, $$|\vec{E}| = \frac{\rho}{3\varepsilon_0} r \quad \text{so} \quad \vec{E} = \frac{\rho}{3\varepsilon_0} r \hat{r}$$

$\vec{E}$ vanishes only at the origin.

<u>Tutorial</u>: <u>SLAC</u> Lightning <u>Strike</u>

- Ask students to share out process for questions i–iii

<u>Clicker</u> <u>Question</u>: Part iv. Does σ affect inside? Outside?

- Discussion of Gauss' Law for charged insulating rod $\Rightarrow$ 

Does it apply? Can it be used?

So Gauss' Law can greatly simplify certain problems, but it also is kind of limited in its utility.

Basically, there are 3 pricipal cases of symmetry.

spherical geometry

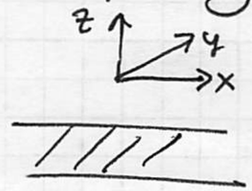
Q is uniform in ϕ, θ but maybe not in r!

Draw a spherical Gaussian surface

cylindrical

Q uniform in θ, z but maybe not in r! (Better be infinite in z!)

Draw a cylindrical "beer can" surface



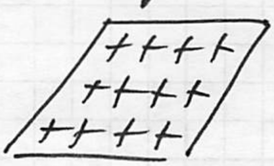
Qs uniform in x,y maybe not z (Better be infinite in x,y)

Draw a "pill box" *

*Example: Let's do the Planar case (illustrate the pillbox)

Consider a sheet with uniform charge density, σ in the x-y plane

What is $\vec{E}(x,y,z)$?

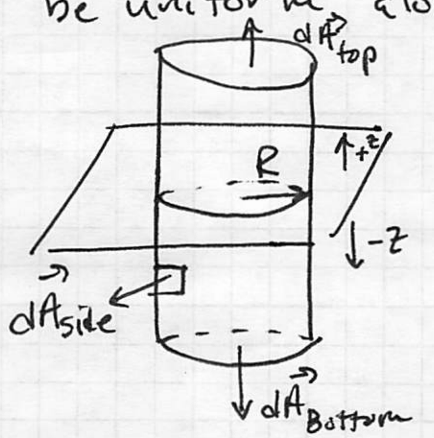


This problem is obviously an idealized situation, but any flat surface will look like this when you are close to it. This will be a useful result!

* Also it demonstrates a deeply important result regarding the boundary between the two sides of the plane (i.e. $\pm z$).

What imaginary surface do we pick?

- Symmetry suggests that $\vec{E}(x,y,z)$ can only point in $\pm\hat{z}$ direction.
  (Do you see why?)
- Symmetry also suggest that $\vec{E}(x,y,z)$ only depends on z. $\vec{E}(x,y,z) = E(z)\,\hat{k}$
  - to see this consider being blindfolded a dropped on the plane somewhere. w/o a measurement tool (a label axis), you can't tell where you are $\Rightarrow \vec{E}$ must not depend on x+y, then!
- These arguments suggest that we should draw a pillbox with faces $\perp$ to the plane or parallel to $\hat{k}$.
- Doing so will make $\vec{E}\cdot d\vec{A}$ simple and $\vec{E}$ will be uniform along surfaces.



So,

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$$

$$\oint_S \vec{E}\cdot d\vec{A} = \int_{top} \vec{E}\cdot d\vec{A}_{top} + \int_{\substack{curvy\\side}} \vec{E}\cdot d\vec{A}_{side} + \int_{bottom} \vec{E}\cdot d\vec{A}_{bottom}$$

$$= \frac{Q_{enc}}{\epsilon_0} = \frac{\sigma\pi R^2}{\epsilon_0}$$

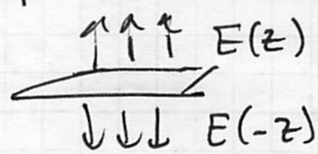
$$\underbrace{\text{top}} + \underbrace{\text{side}} + \underbrace{\text{bottom}} = \frac{\sigma \pi R^2}{\epsilon_0}$$

$$E(z)\pi R^2 + \underbrace{0}_{\vec{E}\cdot d\vec{A}=0} + E(-z)\pi R^2 \underbrace{(-1)}$$

because $d\vec{A}_{\text{bottom}}$ points down

If we look carefully at our picture we notice that

$$E(z) = -E(-z) \implies$$



so

$$E(z)\pi R^2 + E(-z)\pi R^2(-1) = 2E(z)\pi R^2 = \frac{\sigma \pi R^2}{\epsilon_0}$$

this is a critical and yet subtle point.. the left hand side of this equation is the total flux, which is equal to the flux through the top AND bottom. But both are equal, so sum is twice the flux through top.

$$2E(z)\pi R^2 = \frac{\sigma \pi R^2}{\epsilon_0} \qquad E(z) = \frac{\sigma}{2\epsilon_0}$$

independent of z!

so

$$\vec{E}(z) = \begin{cases} \dfrac{\sigma}{2\epsilon_0}\hat{k} & \text{if } z>0 \\ -\dfrac{\sigma}{2\epsilon_0}\hat{k} & \text{if } z<0 \end{cases}$$

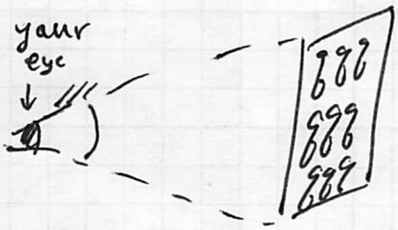
there's a couple things to note about this example.

① $\vec{E}$ is discontinuous at the sheet!

$\Delta E = \sigma/\epsilon_0$ as you move across.

This (as we will see) is "deep" & universal!

② $E(z)$ is constant! Moving away from an infinite sheet has no effect!
(Think: How could you tell how far you are?)

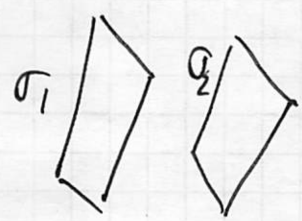


As you move away,
$E_{\text{from each } q} \propto 1/r^2$
But you "see" an area $\propto r^2$
So the effects cancel!

③ R of the imaginary surface dropped out of the solution.

Gauss' Law is universally applicable but not as useful as we might like. Fortunately, we have the superposition principle to help!

For example,



$\vec{E}$ everywhere is due to
$\vec{E}_1 \Rightarrow \sigma_1$ plus
$\vec{E}_2 \Rightarrow \sigma_2$

We solved the sphere and the plane. Think about or look up how you deal with a line charge or an infinitely long cylinder (or cylindrical shell).