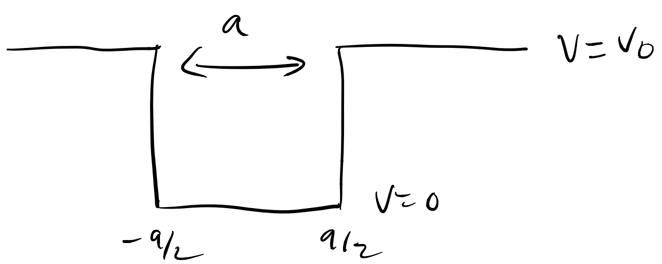
Wrapping up Finite Square Well



$$z = \sqrt{\frac{2mEa^2}{\hbar^2}} \qquad z_0 = \sqrt{\frac{2m(V_0 - E)a^2}{\hbar^2}}$$

$$\boxed{\begin{aligned} z\tan(z) &= \sqrt{z_0^2 - z^2} \\ -z\cot(z) &= \sqrt{z_0^2 - z^2} \end{aligned}}$$

Condition relationship between
V_0 & E (allowed energies)

1. Sketch potential
2. Write down $\hat{H}|E\rangle = E|E\rangle$
   any "regions"
   value of $V(x)$
3. ask question sign V? E?
   Bound states $E < \underline{V_0}$
4. Differentia Equ. signs V_0, E
5. Posit gen. solution
6. Match BCs

Boundary Conditions

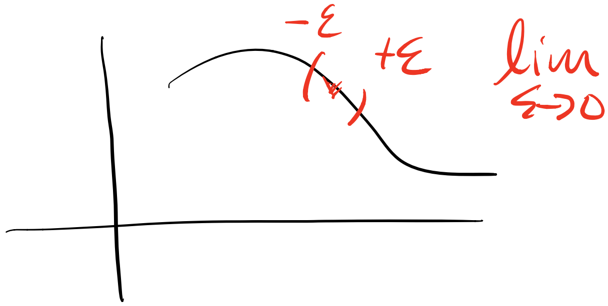
$\psi_E(x)$ is continuous $\quad P(x)$ continuous

$$\int_{-\infty}^{+\infty} P(x)\,dx = 1$$

$\dfrac{d\psi_E(x)}{dx}$ smooth?

continuous provided $V \cancel{\to\infty}$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_E(x) + V(x)\psi_E(x) = E\psi_E(x)$$



$$\lim_{\varepsilon\to 0}\underbrace{\int_{-\varepsilon}^{\varepsilon} -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\varphi_E(x)\,dx}+\int_{-\varepsilon}^{+\varepsilon} V(x)\,\varphi_E(x)\,dx$$

$$= \underbrace{\int_{-\varepsilon}^{+\varepsilon} E\,\varphi_E(x)\,dx}$$

$$\lim_{\varepsilon\to 0} E\int_{-\varepsilon}^{+\varepsilon}\varphi_E(x)\,dx = 0$$

b/c $\varphi_E(x)$ continuos

$$-\frac{\hbar^2}{2m}\int_{-\varepsilon}^{+\varepsilon}\frac{d^2}{dx^2}\varphi_E(x)\,dx = -\int_{-\varepsilon}^{+\varepsilon} V(x)\,\varphi_E(x)\,dx$$

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{+\varepsilon}\frac{d^2}{dx^2}\varphi_E(x)\,dx = \lim_{\varepsilon\to 0}\underbrace{\left.\frac{d\varphi_E}{dx}\right|_{+\varepsilon}-\left.\frac{d\varphi_E}{dx}\right|_{-\varepsilon}}_{\text{Difference } d\varphi/dx}$$

$$\frac{2m}{\hbar^2}\int_{-\varepsilon}^{+\varepsilon} V(x)\,\psi_E(x)\,dx$$

well behaved doesn't blow up $= 0$

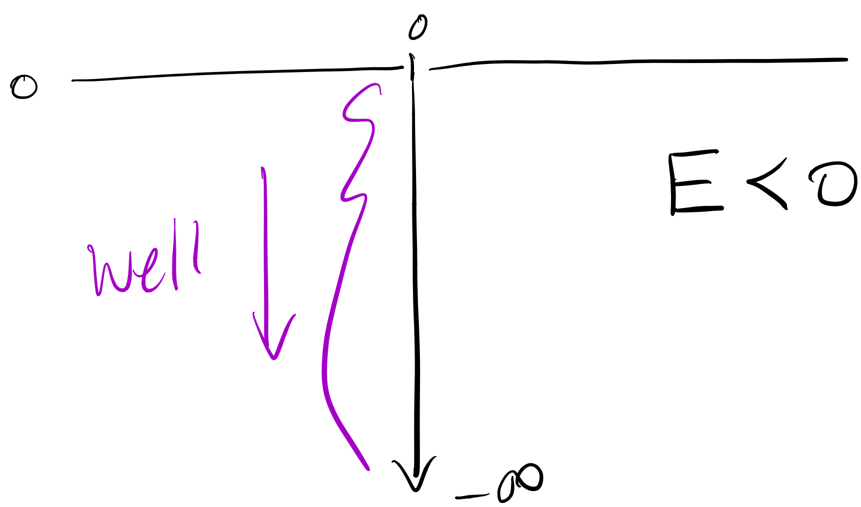
does blow up!

$$\underbrace{\frac{2m}{\hbar^2}\int_{-\varepsilon}^{+\varepsilon} V(x)\,\psi_E(x)\,dx}_{0} = \left.\frac{d\psi_E}{dx}\right|_{\varepsilon} - \left.\frac{d\psi}{dx}\right|_{-\varepsilon}$$

$$V = -\beta\,\delta(x)$$

Example: δ function Potential

$$V(x) = \begin{cases} 0 & x<0 \\ -\beta\delta(x) & x=0 \\ 0 & x>0 \end{cases}$$



$E<0$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_E(x) + V(x)\psi_E(x) = E\psi_E(x)$$

$$q = \sqrt{-\frac{2mE}{\hbar^2}} > 0$$

$x \neq 0$

$$\boxed{\frac{d^2}{dx^2}\psi_E(x) = +q^2\psi_E(x)}$$

$$\psi_E(x) = \begin{cases} A e^{qx} & x < 0 \\ \quad e^{-qx} & \\ B e^{-qx} & x > 0 \\ \quad e^{+qx} & \end{cases}$$

$\rightarrow \psi \not\to \infty$ at $x \to -\infty$

$$\underset{x<0}{\psi_E(0)} = \underset{x>0}{\psi_E(0)} \quad \underline{\text{continuity}}!$$

$$\underset{x<0}{\frac{d\psi_E(x)}{dx}} = \underset{x>0}{\frac{d\psi_E(x)}{dx}} \quad ??$$

$\underline{\underline{\text{No!}}}$

$$V(x) \to -\infty \quad \text{at} \quad x = 0$$

$$\left.\frac{d\psi_E(x)}{dx}\right|_{+\varepsilon} - \left.\frac{d\psi_E(x)}{dx}\right|_{-\varepsilon}$$

$$= \frac{2m}{\hbar^2} \int_{-\varepsilon}^{+\varepsilon} V(x)\, \psi_E(x)\, dx$$

$$V(x) = -\beta\, \delta(x)$$

$$\int_{-\infty}^{\infty} f(x)\, \delta(x)\, dx = f(0)$$

$$\int_{-\infty}^{\infty} f(x)\, \delta(x-a)\, dx = f(a)$$

$$= \frac{2m}{\hbar^2} \int_{-\varepsilon}^{+\varepsilon} V(x)\, \psi_E(x)\, dx$$

$$= -\frac{2m\beta}{\hbar^2} \int_{-\varepsilon}^{\varepsilon} \delta(x)\, \psi_E(x)\, dx$$

$$= \boxed{-\frac{2m\beta}{\hbar^2}\varphi_E(0) = \left.\frac{d\varphi}{dx}\right|_{\varepsilon} - \left.\frac{d\varphi}{dx}\right|_{-\varepsilon}}$$

$$\boxed{\varphi_E(0) = \varphi_E(0)}$$

$$\varphi_E(x) = \begin{cases} Ae^{gx} & x<0 \\ Be^{-gx} & x>0 \end{cases}$$

$\varphi_E(0) = \varphi_E(0) \implies A = B$ symmetric well!

$$\varphi_E(x) = \begin{cases} Ae^{gx} & x<0 \\ Ae^{-gx} & x>0 \end{cases}$$

$$\lim_{\varepsilon\to 0} \left.\frac{d\varphi_E}{dx}\right|_{\varepsilon} - \left.\frac{d\varphi}{dx}\right|_{-\varepsilon} = -\frac{2m\beta}{\hbar^2}\varphi_E(0)$$

$$\lim_{\varepsilon \to 0} \left(-q A e^{-q\varepsilon} - q A e^{-q\varepsilon} \right) = \frac{-2m\beta}{\hbar^2} \psi_E(0)$$

$$-2qA \underbrace{\left(\lim_{\varepsilon \to 0} e^{-q\varepsilon} \right)}_{1 = e^0} = \frac{-2m\beta}{\hbar^2} \psi_E(0)$$

A b/c both $\psi_E(x<0)$ and $\psi_E(x>0)$ give that!

$$-2qA = \frac{-2m\beta}{\hbar^2} A$$

$$q = m\beta/\hbar^2 \quad \text{or} \quad q = \sqrt{\frac{-2mE}{\hbar^2}}$$

$$\boxed{E = -\frac{\hbar^2 q^2}{2m} = -\frac{m\beta^2}{2\hbar^2}}$$

one allowed energy

$$\varphi_E(x) = \begin{cases} Ae^{qx} & x<0 \\ Ae^{-qx} & x>0 \end{cases}$$

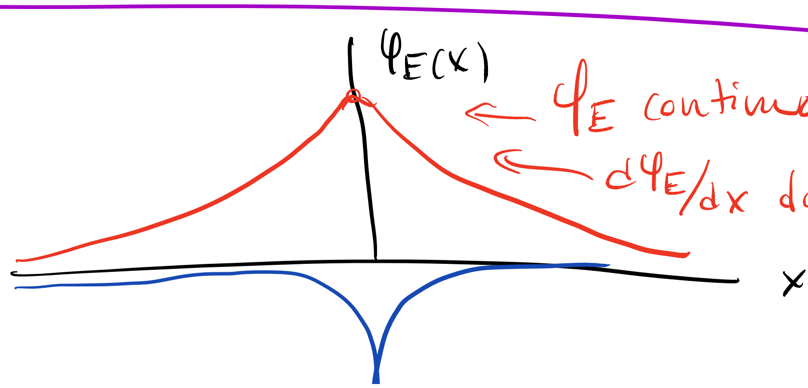
$$\langle \varphi_E | \varphi_E \rangle = \int_{-\infty}^{\infty} \varphi_E^* \varphi_E \, dx = 2\int_0^{\infty} |\varphi_E|^2 dx$$

$$= 2|A|^2 \int_0^{\infty} e^{-2qx} dx = 2|A|^2 \left[\frac{-1}{2q} \left(e^{-2qx} \right) \right]_0^{\infty}$$

$$= 2|A|^2 \left(-\frac{1}{2q} \right) (0-1) = \frac{|A|^2}{q}$$

$$A = |q| = \frac{\sqrt{m\beta}}{\hbar}$$

$$\boxed{\varphi_E(x) = \begin{cases} \sqrt{m\beta}/\hbar \; e^{m\beta x/\hbar^2} & x<0 \\ \sqrt{m\beta}/\hbar \; e^{-m\beta x/\hbar^2} & x>0 \end{cases}}$$



$\leftarrow$ φ_E continuous @ $x=0$

$\leftarrow$ $d\varphi_E/dx$ discontinuous

$$\Big|\; -\beta\,\delta(x)$$