Quantum Harmonic Oscillator

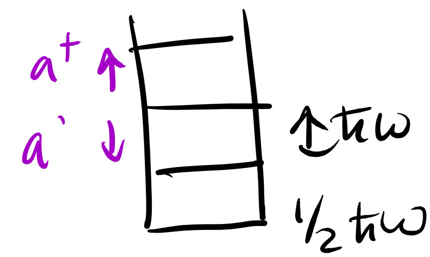
$$V(x) = \frac{1}{2} m\omega^2 x^2$$

$$H = T + V \Rightarrow H|n\rangle = E_n|n\rangle$$

energy eigenstate

$$H|n\rangle = (n+\tfrac{1}{2})\hbar\omega|n\rangle$$

$$E_n = (n+\tfrac{1}{2})\hbar\omega$$



$$a \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\,\hat{p}/m\omega\right) \quad \text{lowering}$$

$$a^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - i\,\hat{p}/m\omega\right) \quad \text{raising}$$

$$a|0\rangle = 0$$

$$\varphi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$$

$$\boxed{a|n\rangle = \sqrt{n}\,|n-1\rangle \qquad |n-1\rangle = \frac{a|n\rangle}{\sqrt{n}}}$$

$$\boxed{a^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad |n+1\rangle = \frac{a^{+}|n\rangle}{\sqrt{n+1}}}$$

From $|0\rangle$ state you can generate all states

$$|1\rangle = \frac{1}{\sqrt{1}} a^{+}|0\rangle$$

$$|2\rangle = \frac{1}{\sqrt{2}} a^{+}|1\rangle = \frac{1}{\sqrt{2\cdot 1}}(a^{+})^{2}|0\rangle$$

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^{+})^{n}\,|0\rangle$$

$$\langle x|n\rangle \doteq \varphi_n(x) = \underbrace{\frac{1}{\sqrt{n!}}}_{\text{normalize}}\underbrace{\left[\sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{\hbar}{m\omega}\frac{d}{dx}\right)\right]^{n}}_{\text{operator}}\underbrace{\varphi_0}_{\text{ground state}}$$

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let $\xi \equiv \sqrt{\frac{m\omega}{\hbar}}\,x$

$$\boxed{\begin{aligned}\varphi_0(x) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\xi^2/2}\\ \varphi_n(x) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}\underbrace{H_n(\xi)}_{\substack{\text{newbit}\\ \text{Hermite}\\ \text{poly}}}e^{-\xi^2/2}\end{aligned}}$$

Hermite Polynomials

$\xi \to y$

$H_0(y) = 1$ $\qquad$ $H_2(y) = 4y^2 - 2$

$H_1(y) = 2y$ $\qquad$ etc.

Representations QHO

QHO energy eigenstates!

$|n\rangle$ ket $\qquad \langle m|n\rangle = \delta_{mn}$

$\langle x|n\rangle \doteq \varphi_n(x)$ position rep $\qquad \int_{-\infty}^{+\infty} \varphi_m^* \varphi_n \, dx = \delta_{mn}$

Matrix Rep (Heisenberg)

Spin:

$|+\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |-\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad S_z \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

S_z basis $\qquad$ diagonal

An operator is diagonal in it's own basis.

$$S_z|+\rangle \doteq \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1\\0\end{pmatrix}$$

$$S_z|+\rangle = \frac{\hbar}{2}|+\rangle$$

Hilbert Space $\rightarrow$ 2D $\qquad \uparrow \;\downarrow$

QHO $\rightarrow$? any # of dimensions

$$H|n\rangle = (n+\tfrac{1}{2})\hbar\omega\,|n\rangle$$

$$n = 0, 1, 2, \ldots\ldots$$

$H \doteq$ 

Inf # of entries?

$E_n = (n+\frac{1}{2})\hbar\omega$ expectation in energy basis only diagonal elements

$$H \doteq \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & \frac{1}{2}\hbar\omega & 0 & 0 & 0 \\ 1 & 0 & \frac{3}{2}\hbar\omega & 0 & 0 \\ 2 & 0 & 0 & \frac{5}{2}\hbar\omega & 0 \\ 3 & 0 & 0 & 0 & \frac{7}{2}\hbar\omega \\ & \vdots & \vdots & \vdots & \vdots \quad \ddots \end{array}$$

$$\underline{\langle 0|H|0\rangle} = \frac{1}{2}\hbar\omega$$

$$\underline{\langle 1|H|0\rangle} = 0 = \langle 0|H|1\rangle$$

$$|0\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad |1\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad |2\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

$$|\psi\rangle = \sum_{i=0}^{\infty} c_i |i\rangle \qquad c_n = \langle n|\psi\rangle$$

$$|\psi\rangle \doteq \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \\ \vdots \end{pmatrix} \qquad \langle n|\psi\rangle = c_n$$

$$(0\,0\,0\,1\,0\,0\,0)\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \\ \vdots \end{pmatrix} = c_n$$

What about operators?

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

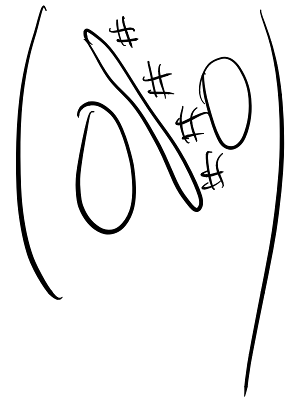
$$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$\langle m|a|n\rangle = \sqrt{n}\,\langle m|n-1\rangle = \sqrt{n}\,\delta_{m,n-1}$$

$$\langle m|a^\dagger|n\rangle = \sqrt{n+1}\,\langle m|n+1\rangle = \sqrt{n+1}\,\delta_{m,n+1}$$

only adjacent states are connected by a, $a^\dagger$

$$
a \doteq
\begin{array}{c|cccccc}
 & 0 & 1 & 2 & 3 & 4 & \cdots \\
\hline
0 & 0 & \sqrt{1} & 0 & 0 & 0 & \\
1 & 0 & 0 & \sqrt{2} & 0 & 0 & \\
2 & 0 & 0 & 0 & \sqrt{3} & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & & \\
4 & 0 & 0 & 0 & 0 & 0 & \\
\vdots & \vdots & \vdots & \vdots & & \ddots &
\end{array}
\Rightarrow
$$



$$
a^\dagger \doteq \begin{pmatrix} 0 & & & & \\ \sqrt{1} & 0 & & 0 & \\ 0 & \sqrt{2} & 0 & & \\ 0 & 0 & \sqrt{3} & 0 & \\ & & & & \ddots \end{pmatrix}
$$

matrix element $\Rightarrow$ $\langle m | a | n \rangle$

$$
\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(a^\dagger + a)
$$

$$\hat{X} \doteq \frac{\hbar}{2m\omega}\left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ \sqrt{1} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix} + \begin{pmatrix} 0 & \sqrt{1} & & 0 \\ & 0 & \sqrt{2} & \\ 0 & & 0 & \sqrt{3} \end{pmatrix}\right)$$

$$= \frac{\hbar}{2m\omega}\begin{pmatrix} 0 & \sqrt{1} & 0 & \cdots \\ \sqrt{1} & 0 & \sqrt{2} & \cdots \\ 0 & \sqrt{2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Time Dependence

$$|\psi(0)\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$$

Time evol → simple E_n, $|n\rangle$

$$|n\rangle e^{-iE_n t/\hbar} = |n(t)\rangle \quad \underline{\underline{\text{S.E.}}}$$

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} c_n e^{-iE_n t/\hbar} |n\rangle$$

$$\boxed{|\psi(t)\rangle = e^{-i\omega t/2} \sum_{n=0}^{\infty} c_n e^{-in\omega t} |n\rangle}$$

Time Evol QHO.

Example:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|\psi(t)\rangle = e^{-i\omega t/2}\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}e^{-i\omega t}|1\rangle\right)$$

$$\left.\begin{aligned} P_0 &= |\langle 0|\psi(t)\rangle|^2 \\ P_1 &= |\langle 1|\psi(t)\rangle|^2 \end{aligned}\right\}$$

all the same

$$\left|\langle 0|\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}e^{-i\omega t}|1\rangle\right)\right|^2$$

$$P_0 = \left|\langle 0| \tfrac{1}{\sqrt{2}} |0\rangle\right|^2 = \frac{1}{2}$$

$$P_1 = 1 - P_0 = \frac{1}{2}$$

$$\underline{P(x,t)} = |\underline{\langle x|\psi(t)\rangle}|^2$$

$$= |\psi(x,t)|^2 \qquad \begin{array}{l}\langle x|0\rangle \doteq \varphi_0 \\ \langle x|1\rangle \doteq \varphi_1\end{array}$$

$$= \left|\langle x|\, \underline{e^{-i\omega t/2}} \left[\tfrac{1}{\sqrt{2}}|\underline{0}\rangle + \tfrac{1}{\sqrt{2}} e^{-i\omega t}|1\rangle\right]\right|^2$$

$$= \frac{1}{2}\left|\varphi_0(x) + e^{-i\omega t}\varphi_1(x)\right|^2$$

$$= \frac{1}{2}\left(\varphi_0^2(x) + \varphi_1^2(x) + \underline{\varphi_0\varphi_1^* e^{+i\omega t}} + \underline{\varphi_1}\varphi_0^* e^{-i\omega t}\right)$$

$$\underline{\varphi_n(x) \Rightarrow \text{Real}}$$

$$\varphi_n(x) = \underline{\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}} \underline{\frac{1}{\sqrt{2^n n!}}} \underline{H_n(\xi)} \underline{e^{-\xi^2/2}}$$

$$\xi = \sqrt{\frac{m\omega}{\hbar}}\, x$$

$$\psi_0 \psi_1 e^{+i\omega t} + \psi_0 \psi_1 e^{-i\omega t}$$

$$= 2\psi_0 \psi_1 \cos(\omega t) \quad \text{Real}$$

$$P(x,t) = \frac{1}{2}\left(\psi_0^2 + \psi_1^2 + 2\psi_0\psi_1 \cos(\omega t) \right)$$

$\langle \hat{x} \rangle \implies a^+ \quad a$ operations

$$\langle \psi(t) | \hat{x} | \psi(t) \rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \langle \psi(t) | a^+ + a | \psi(t) \rangle$$

$\langle 0 | a^+ | 1 \rangle \quad \langle 0 | a | 1 \rangle$ etc.

$$= \sqrt{\frac{\hbar}{2m\omega}} \left(\frac{1}{\sqrt{2}} \langle 0| + \frac{1}{\sqrt{2}} e^{+i\omega t} \langle 1| \right) \left(a^\dagger + a \right) \left(\frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} e^{-i\omega t} |1\rangle \right)$$

$$\sqrt{\frac{\hbar}{2m\omega}} \, \frac{1}{2} \Big(\langle 0| a^\dagger + a |0\rangle + e^{i\omega t} \langle 1| a^\dagger + a |0\rangle + e^{-i\omega t} \langle 0| a^\dagger + a |1\rangle + \langle 1| a^\dagger + a |1\rangle \Big)$$

$$\langle 0| a^\dagger |0\rangle + \langle 0| \underbrace{a |0\rangle}_{=0}$$

$$\langle 0 | 1 \rangle = 0$$

$$\langle 1 | a^\dagger + a | 0 \rangle = \langle 1 | a^\dagger | 0 \rangle + \langle 1 | \underbrace{a | 0 \rangle}_{0}$$

$$= \sqrt{1} \langle 1 | 1 \rangle = \sqrt{1} = 1$$

$$\langle \hat{x} \rangle = \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left(e^{-i\omega t} + e^{+i\omega t}\right)$$

$$\langle \hat{x} \rangle = \sqrt{\frac{\hbar}{2m\omega}} \cos(\omega t)$$