Quantum Harmonic Oscillator

$\Rightarrow$ Classical SHO

Classical Phys $\longrightarrow$ Quantum Phys.

Hamiltonian $\rightarrow$ $H = T + V$ classical physics

$$\underbrace{\frac{\partial H}{\partial x} \qquad \frac{\partial H}{\partial p}} \Rightarrow \text{equations of motion}$$

$$f(x, \dot{x}, \ddot{x}) = 0 \longrightarrow \underset{\text{trajectory}}{x(t)}$$

Classical Ham.

$\hookrightarrow$ $\boxed{\text{operator framework}}$ $\longrightarrow$ Quantum Ham.

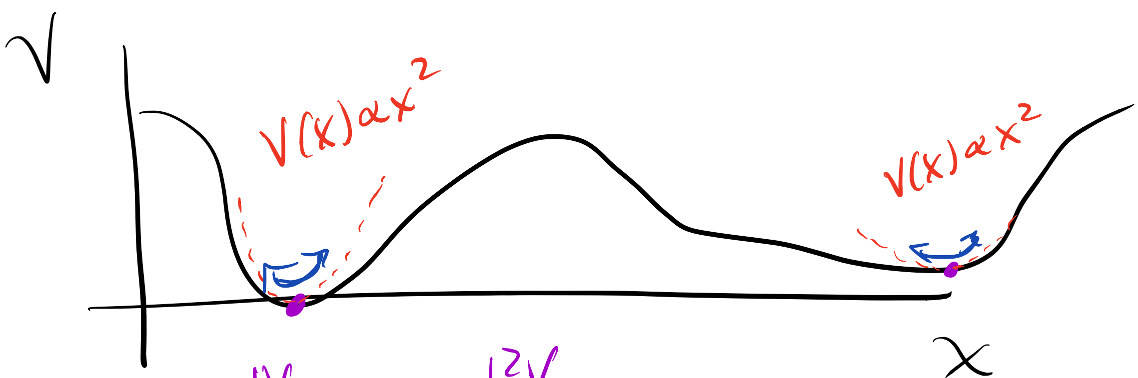
$\hat{x}, \hat{p}$

$$H|\underset{|E\rangle}{\psi}\rangle = E\underset{|E\rangle}{|\psi\rangle}$$

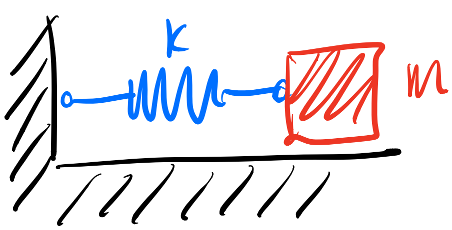
$$\hat{H}|E\rangle = E|E\rangle$$

Time evol $\rightarrow$ $|E\rangle e^{-iEt/\hbar}$

Classical SHO first

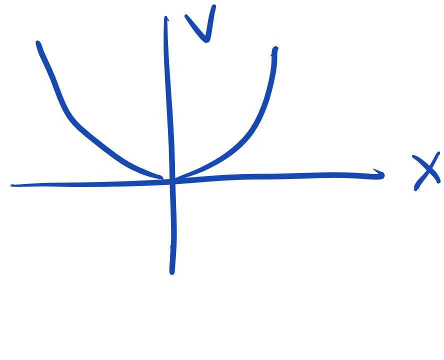


$$\frac{dV}{dx} = 0 \qquad \frac{d^2V}{dx^2} > 0$$



$$F = -kx$$

$$V(x) = \frac{1}{2}kx^2$$



$$V(x) \simeq V(x_0) + \frac{dV}{dx}\bigg|_{x_0}(x-x_0) + \frac{1}{2}\frac{d^2V}{dx^2}\bigg|_{x_0}(x-x_0)^2$$

B/c x_0 equilibrium pt. & minimum

$$\frac{dV}{dx}\bigg|_{x_0} = 0 \qquad \frac{d^2V}{dx^2}\bigg|_{x_0} > 0$$

$$V(x) - V(x_0) = +\frac{1}{2}\frac{d^2V}{dx^2}\bigg|_{x_0}(x-x_0)^2$$

$$V(x) \simeq \frac{1}{2}kx^2 \Rightarrow F = -kx$$

$$m\ddot{x} = -kx \qquad \boxed{\ddot{x} = -\frac{k}{m}x}$$

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}\,t + \phi\right)$$

$$\ddot{x} = -\frac{k}{m}x = -\omega^2 x$$

$$\omega^2 \equiv k/m$$

frequency more useful in QM

Classical

$$H = T + V = \frac{p^2}{2m} + V(x)$$

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2 \quad \boxed{k = m\omega^2}$$

$$\boxed{H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2}$$

QUANTUM

$\downarrow$ $\boxed{\text{operator}}$ $\downarrow$

$$\boxed{\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 \hat{x}^2}$$

Quantum
Harmonic
Osc.
Ham.

Quantum Harmonic Oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

position rep. $\hat{p} = -i\hbar \frac{d}{dx}$ $\hat{x} = x$

$\psi_E(x)$

momentum rep. $\hat{p} = p$ $\hat{x} = i\hbar \frac{d}{dp}$

$\phi_E(p)$

(1)

$$H|E\rangle = E|E\rangle$$

(2)

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_E(x) + \frac{1}{2}m\omega^2 x^2 \psi_E(x) = E\,\psi_E(x)$$

Operator Method

- seeks to factor $\hat{H}$ using operators
- look at how operators act on kets

↳ energy spectrum ⇐

↳ E_n ?

$$\left(\underbrace{\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2}_{\text{sum of two squares}}\right)\varphi_E = E\varphi_E$$

$$(a^2 - b^2) = (a+b)(a-b) \quad \text{Alg. II}$$

$$(u^2+v^2) = (u^2 \underbrace{- iuv + ivu}_{\text{assertion: } uv = vu \text{ commute}} + v^2)$$

$[u, v] = 0$

$$(u^2+v^2) = (u+iv)(u-iv) \quad \text{Alg II \& complex \#s.}$$

$$[\text{operator } 1, \text{operator } 2] \neq 0$$

$$[\hat{x}, \hat{p}] = i\hbar$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

energy

$$= \frac{1}{2}m\omega^2\left[\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2}\right]$$

$$= \frac{\hbar\omega}{\hbar\omega}\left(\frac{1}{2}m\omega^2\right)\left[\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2}\right]$$

$$= \underbrace{(\hbar\omega)}_{\text{energy}}\underbrace{\left[\frac{m\omega}{2\hbar}\left\{\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2}\right\}\right]}_{\text{unit less}}$$

Introduce raising/lowering (ladder) operators

$$a \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)$$

Non-Hermitian operators

$$a^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}^\dagger - \frac{i\hat{p}^\dagger}{m\omega}\right) = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)$$

Hamiltonian in terms of a & $a^\dagger$

keep order straight

$$a^\dagger a = \frac{m\omega}{2\hbar}\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right)\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)$$

$$= \frac{m\omega}{2\hbar}\left(\hat{x}\hat{x} + \frac{i\hat{x}\hat{p}}{m\omega} - \frac{i\hat{p}\hat{x}}{m\omega} + \frac{\hat{p}\hat{p}}{m^2\omega^2}\right)$$

$$= \frac{m\omega}{2\hbar}\left(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} + \frac{i}{m\omega}\left(\hat{x}\hat{p} - \hat{p}\hat{x}\right)\right)$$

$[\hat{x}, \hat{p}]$

$$a^\dagger a = \frac{m\omega}{2\hbar}\left(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} + \frac{i}{m\omega}\underbrace{[\hat{x}, \hat{p}]}_{i\hbar}\right)$$

$$\boxed{a^\dagger a = \frac{m\omega}{2\hbar}\left(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2}\right) - \frac{1}{2}}$$

$$\hat{H} = \hbar\omega \left\{ \frac{m\omega}{2\hbar}\left(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2}\right)\right\}$$

$$\boxed{\hat{H} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)}$$

QHO Ham. in terms $a^\dagger$ & a.

$$aa^\dagger = \frac{m\omega}{2\hbar}\left(\hat{x} + \frac{\hat{p}^2}{m^2\omega^2}\right) + \frac{1}{2}$$

$$\boxed{[a, a^\dagger] = aa^\dagger - a^\dagger a = 1}$$

$$H|E\rangle = E|E\rangle \ ?$$

$$[H, a] = Ha - aH$$

$$= \hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right)a - a\hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right)$$

$$= \hbar\omega\left(a^\dagger a a - a a^\dagger a\right)$$

$$aa^\dagger = 1 + a^\dagger a$$

$$[H, a] = \hbar\omega(a^\dagger a a - (1 + a^\dagger a)a)$$

$$= \hbar\omega(\cancel{a^\dagger a a} - a - \cancel{a^\dagger a a})$$

$$\boxed{[H, a] = -\hbar\omega a} \qquad \boxed{[H, a^\dagger] = +\hbar\omega a^\dagger}$$

$$H(a|E\rangle) = Ha|E\rangle$$

$$Ha = aH - \hbar\omega a \qquad [H, a] = -\hbar\omega a$$

$$aH|E\rangle - \hbar\omega a|E\rangle \qquad H|E\rangle = E|E\rangle$$

$$= aE|E\rangle - \hbar\omega a|E\rangle$$

$$= (E - \hbar\omega)(a|E\rangle) = H(a|E\rangle)$$

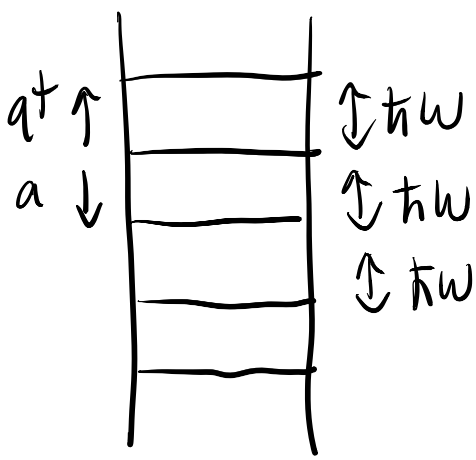
$a|E\rangle$ energy eigenstate w/

eigenvalue $E-\hbar\omega$

$a^{\dagger}$?

$$Ha^{\dagger}|E\rangle = (E+\hbar\omega)(a^{\dagger}|E\rangle)$$

$a^{\dagger}|E\rangle$ energy eigenstate w/ eigenvalue $E+\hbar\omega$



energy spectrum

lowest energy state?

$$a|E_{ground}\rangle = 0$$

ladder term. cond.

$$\hat{H}|E_{ground}\rangle = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)|E_{ground}\rangle$$

$$= \hbar\omega \underbrace{a^\dagger a|E_{ground}\rangle}_{=0} + \frac{1}{2}\hbar\omega|E_{ground}\rangle$$

$$\hat{H}|E_{ground}\rangle = \frac{1}{2}\hbar\omega|E_{ground}\rangle$$

$$\boxed{E_n = \left(n+\frac{1}{2}\right)\hbar\omega \qquad n = 0, 1, 2, \ldots}$$