$$\frac{d^2R}{d\rho^2} + \frac{2}{\rho}\frac{\partial R}{\partial \rho} + \left[-\gamma^2 + \frac{2}{\rho} - \frac{\ell(\ell+1)}{\rho^2}\right]R = 0$$

Asymptotic solutions

$\rho \to \infty$; $\rho \to 0$

solve ⟷ solve

① $\rho \to \infty$ $\qquad \dfrac{d^2R}{d\rho^2} - \gamma^2 R \simeq 0$

$R(\rho) \sim e^{\pm\gamma\rho}$ $\qquad$ ~~$e^{+\gamma\rho}$~~ blows up $\rho \to \infty$

$R(\rho) \sim e^{-\gamma\rho}$

ρ^{q-2}

② $\rho \to 0$ $\qquad \dfrac{d^2R}{d\rho^2} + \dfrac{2}{\rho}\dfrac{dR}{d\rho} - \dfrac{\ell(\ell+1)}{\rho^2}R \simeq 0$

$R(\rho) = \rho^{q}$

$$\frac{dR}{dq} = q\rho^{q-1} \qquad \frac{d^2R}{dq} = q(q-1)\rho^{q-2}$$

$$q(q-1)\rho^{q-2} + \frac{2}{\rho}q\rho^{q-1} - \frac{l(l+1)}{\rho^2}\rho^{q} = 0$$

$$q(q-1)\rho^{q-2} + 2q\rho^{q-2} - l(l+1)\rho^{q-2} = 0$$

$$\boxed{q(q-1) + 2q - l(l+1) = 0}$$

~~$\rho^{q-2} = 0$~~

$$q = l \quad \text{or} \quad q = -(l+1)$$

$R(\rho) \sim \rho^{l}$ ✓ ($\rho \to \infty$) ~~$R(\rho) \sim \rho^{-(l+1)}$~~ ($\rho \to 0$, blows up)

$$R(\rho) \sim e^{-\gamma\rho}$$

$$R(\rho) \sim \rho^{l} e^{-\gamma\rho} \qquad \rho \to 0, \quad \rho \to \infty$$

③ Intermediate Values $f(\rho)$

$$R(\rho) = \rho^{l} e^{-\gamma\rho} f(\rho)$$

Assume → find $f(\rho)$

$$\rho \frac{d^2 f}{d\rho^2} + 2(l+1-\gamma\rho)\frac{df}{d\rho} + 2(1-\gamma-\gamma l) f(\rho) = 0$$

$$f(\rho) = \sum_{j=0}^{\infty} c_j \rho^j \leftarrow$$

$$c_{j+1} = \frac{2\gamma(j+l+1)-2}{(j+1)(j+2l+2)} c_j$$

$c_0 \Rightarrow \langle\psi|\psi\rangle = 1$

$j \to \infty$ lots of terms!

$$c_{j+1} \simeq \frac{\gamma}{j} c_j \Rightarrow f(\rho) \simeq e^{2\gamma\rho}$$

$$R(\rho) \simeq \rho^{l} e^{-\gamma\rho} e^{2\gamma\rho} = \rho^{l} e^{\gamma\rho}$$

grows $\rho \to \infty$

Argument: j terminates

$$2\gamma(j_{max} + l + 1) - 2 = 0$$

j_{max} integer

l ang. mom. Q# integer

$$n \equiv j_{max} + l + 1 \Leftarrow$$ principal QUANTUM #

$j = 0$, $l = 0$ $\Rightarrow n = 1, 2, 3, \ldots$

$$2\gamma n - 2 = 0$$

$$\gamma = 1/n$$

$$-\gamma^2 \equiv E \Big/ \frac{\hbar^2}{2\mu a^2}$$

$$E_n = -\frac{1}{2n^2}\left(\frac{Ze^2}{4\pi\varepsilon_0}\right)\frac{\mu}{\hbar^2} \qquad n = 1, 2, \ldots$$

eigen energies Hydrogen like atom

Ze •   • −e

---

3 quantum numbers

$n = 1, 2, 3, \ldots,$ shell # orbital #

$\ell = 0, 1, 2, 3, \ldots$ ang mom #

$m = -\ell, \ldots, 0, \ldots +\ell$ projection ang mom.

$|n\ell m\rangle$ eigenstates of Hydrogen like atoms

$$|n\ell m\rangle \doteq \psi_{n\ell m}(r, \theta, \phi) = R_{n\ell}(r)\, Y_\ell^m(\theta, \phi)$$