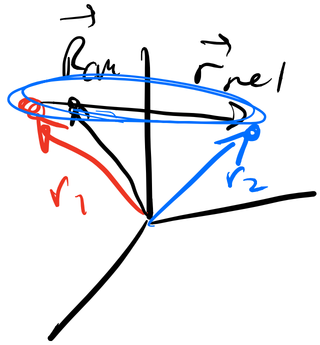
Reminder:



$$H|E\rangle = E|E\rangle$$

$$\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(r)\,|E\rangle$$
$$= E|E\rangle$$

$$\frac{p_{tot}^2}{2M} + \frac{p_{rel}^2}{2\mu} + V(r)\,|E\rangle = E|E\rangle$$

$$H_{cm}|\psi\rangle = E_{cm}|\psi\rangle$$

$$H_{rel}|\psi\rangle = E_{bound}|\psi\rangle$$

$$H_{rel} = \frac{p_{rel}^2}{2\mu} + V(r)$$

central potential

Spherical Coordinates

$\psi(r,\theta,\phi)$; $V(r)$

$$\psi(r,\theta,\phi) = \underline{R(r)}\,\underbrace{\Theta(\theta)\,\Phi(\phi)}$$

spherical harmonics $\Leftarrow Y_l^m(\theta,\phi)$

Radial Eqn.

$A = l(l+1)$ ← takes information ang. mom. state.

$$\boxed{\left[\frac{-\hbar^2}{2\mu r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) + V(r) + l(l+1)\frac{\hbar^2}{2\mu r^2}\right]R(r) = ER(r)}$$

Eigenvalue Eqn

1st term $\frac{d^2}{dr^2}$

2nd term $V(r)$ depends on loc.

3rd term ang. mom. depends on loc

Effective Potential

potential : $\vec{F} \propto -\nabla V$

$V(r)$ not v^2, a etc.

"effective" lump all terms that only depend on loc.

$$V_{eff}(r) = V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2} \quad \text{(Classical Mech)}$$

$V(r)$: Coulomb attractive (-)

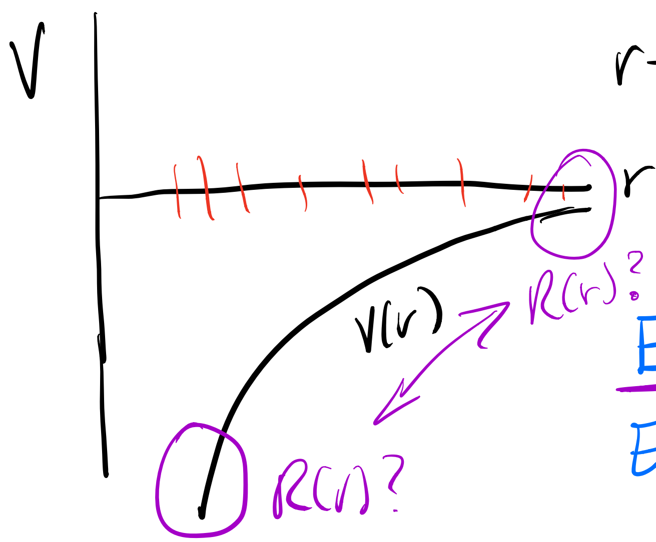
$\frac{l(l+1)\hbar^2}{2\mu r^2}$: repulsive interaction

Specific Potential

$$\boxed{V(r) = -\, Ze^2/4\pi\varepsilon_0 r \quad \text{potential of choice}}$$

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \frac{2\mu}{\hbar^2}\left[E + \frac{Ze^2}{4\pi\varepsilon_0 r} - \frac{\hbar^2 l(l+1)}{2\mu r^2}\right]R = 0$$

Diffy Q to solve for R.



$r \to \infty;\ V(r) \to 0$

well defined zero.

$E < 0$ bound states

$E > 0$ unbound states

New theoretical: seeking asymptotic solutions

"non dimensionalizing" Diffy Q

$\rho \equiv r/a$ dimensionless length

a: dimensions, length unknown.

$R(r) \to R(\rho)$

Replacement:

$$\rho = r/a \longrightarrow r = \rho a$$

$$\frac{d}{dr} = \frac{d\rho}{dr}\frac{d}{d\rho} = \frac{1}{a}\frac{d}{d\rho} \qquad \frac{d^2}{dr^2} = \frac{1}{a^2}\frac{d^2}{d\rho^2}$$

$$\frac{1}{a^2}\frac{d^2R}{d\rho^2} + \frac{1}{a^2}\frac{2}{\rho}\frac{dR}{d\rho} + \frac{2\mu}{\hbar^2}\left[E + \frac{Ze^2}{4\pi\varepsilon_0 a\rho} - \frac{\hbar^2 \ell(\ell+1)}{2\mu a^2\rho^2}\right]R = 0$$

$$\frac{d^2R}{d\rho^2} + \frac{2}{\rho}\frac{dR}{d\rho} + \left[\frac{2\mu a^2}{\hbar^2}E + \left(\frac{\mu Z e^2}{4\pi\varepsilon_0\hbar}\right)\frac{2a}{\rho} - \frac{\ell(\ell+1)}{\rho^2}\right]R = 0$$

dimensionless

$$\frac{\mu Z e^2}{4\pi\varepsilon_0\hbar^2} \sim 1/\text{length}$$

$$\boxed{a \equiv \frac{4\pi\varepsilon_0\hbar^2}{\mu Z e^2} \qquad \text{characteristic length scale}}$$

$$\frac{2\mu a^2}{\hbar^2} \sim 1/\text{Energy}$$

$\frac{\hbar^2}{2\mu a^2}$ characteristic energy scale

$E/(\hbar^2/2\mu a^2)$ unitless

bound states < 0

$$\boxed{-\gamma^2 \equiv \frac{E}{(\hbar^2/2\mu a^2)} < 0}$$

$$\boxed{\frac{d^2R}{d\rho^2} + \frac{2}{\rho}\frac{dR}{d\rho} + \left[-\gamma^2 + \frac{2}{\rho} - \frac{l(l+1)}{\rho^2}\right]R = 0}$$

Solving

1. solve $\rho \to \infty$ (Diffy approx)
2. solve $\rho \to 0$ (Diffy Q approx)
3. $\Rightarrow$ match solutions intermediate

1. $\rho \to \infty$

$$\frac{d^2R}{d\rho^2} + \underbrace{\underbrace{\frac{2}{\rho}}_{\to 0}\frac{dR}{d\rho}}_{\to 0} + \left[-\gamma^2 + \underbrace{\frac{2}{\rho}}_{\to 0} - \underbrace{\frac{\ell(\ell+1)}{\rho^2}}_{\to 0}\right]R = 0$$

$$\frac{d^2R}{d\rho^2} - \gamma^2 R = 0$$

$$R(\rho) \sim e^{\pm\gamma\rho}$$

$$\rho \to \infty \qquad e^{+\gamma\rho} \to \infty \quad \times$$

$$R(\rho) \sim e^{-\gamma\rho}$$

2. $\rho \to 0$

$$\frac{d^2R}{d\rho^2} + \frac{2}{\rho}\frac{dR}{d\rho} + \overbrace{\left[\underbrace{-\gamma^2}_{\text{nominal}} + \underbrace{\frac{2}{\rho}}_{\text{big}} - \underbrace{\frac{\ell(\ell+1)}{\rho^2}}_{\to\ \text{dominates even bigger}}\right]}^{\text{dimensionless}}R = 0$$

$$\underbrace{\frac{d^2R}{d\rho^2}}_{\rho^{g-2}} + \underbrace{\frac{2}{\rho}\frac{dR}{d\rho}}_{\rho^{g-2}} - \frac{\ell(\ell+1)}{\rho^2}R = 0$$

ρ^{g-2}

$R(\rho) = \rho^g$

polynomial

$R(\rho) = C\rho^g$

$R(\rho) = \rho^g$

$R(\rho) = A\rho^g + B\rho^{g-1} + \ldots$