In spin systems we we able to write spin operators & kets in a matrix representation

$$|+\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |-\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad S_z \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

etc.

For the QHO the space is not finite like for spin 1/2 & spin 1 systems it is infinite as $|n\rangle$ $n = 0, 1, 2, 3, \ldots$

and $$H|n\rangle = (n+\tfrac{1}{2})\hbar\omega|n\rangle$$

this seems like a problem, but we can remeber that the matrix representation of an operator is diagonal in it's own basis (see above, spin 1/2 in S_z basis)

as we are using the energy basis $H|n\rangle = (n+\frac{1}{2})\hbar\omega|n\rangle$

The Hamiltonian can thus be written, 2

$$H \doteq \begin{pmatrix} \frac{1}{2}\hbar\omega & 0 & 0 & 0 & \cdots \\ 0 & \frac{3}{2}\hbar\omega & 0 & 0 & \cdots \\ 0 & 0 & \frac{5}{2}\hbar\omega & 0 & \cdots \\ 0 & 0 & 0 & \frac{7}{2}\hbar\omega & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and then the eigenstates are very simple,

$$|0\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \quad |1\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \quad |2\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \text{ etc.}$$

A general ket will be a superposition of basis kets, $|\psi\rangle = \sum_{i=0}^{\infty} c_i |i\rangle$ with $c_n = \langle n|\psi\rangle$

so,

$$|\psi\rangle \doteq \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix}$$

What about the operators?

Recall,

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$
$$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

We can find the matrix elements by projecting onto a different eigenstate,

$$\langle m|a|n\rangle = \langle m|\sqrt{n}\,|n-1\rangle = \sqrt{n}\,\delta_{m,n-1}$$
$$\langle m|a^\dagger|n\rangle = \langle m|\sqrt{n+1}\,|n+1\rangle = \sqrt{n+1}\,\delta_{m,n+1}$$

So adjacent states are connected,

$$a \doteq \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$a^\dagger \doteq \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

other operators can be constructed from a & $a^\dagger$