In our analysis of the QHO we have (1)
so far avoided finding the position representation,
$\psi_n(x)$. Instead we have shown,

$$H|n\rangle = E_n|n\rangle = (n+\tfrac{1}{2}\hbar\omega)|n\rangle$$

$$\langle n|n\rangle = 1 \quad \text{and} \quad \langle m|n\rangle = \delta_{mn}$$

using a operator method with

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega(a^\dagger a + \tfrac{1}{2})$$

with

$$a \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right)$$

and

$$a^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right)$$

Typically we would try to solve our
eigenvalue equation for the eigenstates,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_n}{dx^2} + \frac{1}{2}m\omega^2x^2\psi_n = E_n\psi_n$$

But, we have a simpler way. 2

B/c a & $a^\dagger$ act to "lower" and "raise" states we can find ψ_0 and just raise it repeatedly to find $\psi_{n>0}$.

$$a|0\rangle = 0 \longrightarrow \langle x|0\rangle = \psi_0(x)$$

$$a\psi_0(x) = 0$$

$$\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right)\psi_0(x) = 0$$

$$\sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{\hbar}{m\omega}\frac{d}{dx}\right)\psi_0(x) = 0$$

$$\boxed{\frac{d\psi_0(x)}{dx} = -\frac{m\omega}{\hbar}x\psi_0(x) \quad \text{Diffy Q for } \psi_0(x)}$$

The derivative gives back the function times x so we try an ansatz,

3

$$\varphi_0(x) = Ae^{-\alpha x^2} \quad (\text{a Gaussian})$$

$$\frac{d\varphi_0(x)}{dx} = -2\alpha x Ae^{-\alpha x^2}$$

$$-2\alpha \cancel{x} \cancel{Ae^{-\alpha x^2}} = -\frac{m\omega}{\hbar} \cancel{x} \cancel{Ae^{-\alpha x^2}}$$

$$\alpha = \frac{m\omega}{2\hbar}! \quad \text{So} \quad \boxed{\varphi_0(x) = Ae^{-m\omega x^2/2\hbar}}$$

We still need to find A, we will use normalization.

$$\langle 0|0\rangle = 1$$

$$1 = \int_{-\infty}^{+\infty} |A|^2 e^{-m\omega x^2/\hbar}\, dx = 2|A|^2 \int_0^{\infty} e^{-m\omega x^2/\hbar}\, dx$$

$$= 2|A|^2 \left[\frac{\sqrt{\pi}}{2}\sqrt{\frac{\hbar}{m\omega}}\right] = |A|^2 \sqrt{\frac{\pi\hbar}{m\omega}} = 1$$

$$\boxed{|A| = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}}$$

So,

$$\boxed{\langle x|0\rangle \doteq \varphi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}}$$

4

Beyond the Ground State?

$a^+|0\rangle \propto |1\rangle$ the raising operator will give us back something proportional to the next state

But! No guarantee it is normalized!

Let's see how to handle this normalization issue.

First note that,

$$a^+a|n\rangle = n|n\rangle$$

so let's look at the state,

$$a|n\rangle$$

* Some Books call $a^+a = N$ the number operator where $N|n\rangle = n|n\rangle$

If we compute the norm of $a|n\rangle$ we (5)
have,

$$\left| a|n\rangle \right|^2 = \left(\langle n|a^\dagger\right)\left(a|n\rangle\right) = \langle n|a^\dagger a|n\rangle$$

$$= \langle n|n|n\rangle = n\langle n|n\rangle = n$$

That is the norm of $a|n\rangle$ is equal to n.

We know $a|n\rangle$ is connected to $|n-1\rangle$, but what is the issue with normalization?

$a|n\rangle \propto |n-1\rangle$ assume a constant of proportionality, c, so that

$$a|n\rangle = c\,|n-1\rangle$$

so that

$$\left| a|n\rangle \right|^2 = \left| c\,|n-1\rangle \right|^2$$

$$n = (\langle n-1| c)(c|n-1\rangle) = \langle n-1||c|^2|n-1\rangle$$

$$= |c|^2 \langle n-1|n-1\rangle = |c|^2$$

So $c = \sqrt{n}$ chosen to be real & positive

$$\boxed{\begin{array}{l}\text{The Lowering Operator} \\ \quad a|n\rangle = \sqrt{n}\,|n-1\rangle\end{array}}$$

Let's check $a^\dagger$,

$$|a^\dagger|n\rangle|^2 = (\langle n|a)(a^\dagger|n\rangle) = \langle n|aa^\dagger|n\rangle$$

Note: $[a, a^\dagger] = aa^\dagger - a^\dagger a = 1$

So $aa^\dagger = 1 + a^\dagger a$

$$|a^\dagger|n\rangle|^2 = \langle n|\,1 + a^\dagger a\,|n\rangle$$

$$= \langle n|1|n\rangle + \langle n|a^\dagger a|n\rangle$$

$$= \langle n|1|n\rangle + \langle n|n|n\rangle$$

$$= 1\langle n|n\rangle + n\langle n|n\rangle$$

$$|a^{+}|n\rangle|^{2} = n+1$$

So we again setup the operator equ.

$$a^{+}|n\rangle = c\,|n+1\rangle$$

$$|a^{+}|n\rangle|^{2} = n+1 = |c|^{2} \Rightarrow c = \sqrt{n+1}$$

So

The Raising Operator

$$\boxed{a^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle}$$

Thus to get the normalized state,

$$\boxed{|n+1\rangle = \frac{a^{+}|n\rangle}{\sqrt{n+1}}}$$

8

We can see a pattern,

$$|1\rangle = \frac{1}{\sqrt{1}} a^{+}|0\rangle$$

$$|2\rangle = \frac{1}{\sqrt{2}} a^{+}|1\rangle = \frac{1}{\sqrt{2\cdot 1}}(a^{+})^{2}|0\rangle$$

$$|3\rangle = \frac{1}{\sqrt{3}} a^{+}|2\rangle = \frac{1}{\sqrt{3\cdot 2\cdot 1}}(a^{+})^{3}|0\rangle$$

or,

$$\boxed{|n\rangle = \frac{1}{\sqrt{n!}}(a^{+})^{n}|0\rangle \quad \text{gets any } |n\rangle}$$

In the spatial basis this is,

$$\psi_n(x) = \frac{1}{\sqrt{n!}}\left[\sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{\hbar}{m\omega}\frac{d}{dx}\right)\right]^{n}\psi_0(x)$$

It turns out this position basis wave function can be written using the <u>Hermite Polynomials</u>!

9

let $\xi \equiv \sqrt{\frac{m\omega}{\hbar}}\, x$ then,

$$\varphi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\xi^2/2}$$

and

$$\varphi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi)\, e^{-\xi^2/2}$$

where H_n are the Hermite Polynomials

$H_n(\xi)$ is tabulated in most QM Books.

$$H_0(\xi) = 1$$

$$H_1(\xi) = 2\xi$$

$$H_2(\xi) = 4\xi^2 - 2 \quad \text{etc.}$$