We have looked into three systems - all using some form of 1

$\hat{H} = \hat{T} + \hat{V}$ to set up $\hat{H}|E\rangle = E|E\rangle$

and then investigate the time evolution energy eigenstates and their superposition with $e^{-iEt/\hbar}$ for each eigenstate.

We now turn to another system for the same kind of work → find energy eigenstates
↳ investigate evolution

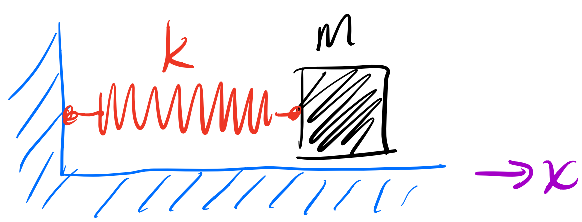
This new system the Quantum harmonic oscillator has a classical analog, the simple harmonic oscillator, which you have seen many times.

We aim to form the Hamiltonian for the QHO and we will do so, as we have before, forming the classical Hamiltonian and then replacing p & x with operators $\hat{p}$ & $\hat{x}$.

Before we do that let's remind ourselves of the classical SHO.

This model (SHO) appears in physics in many places. $\rightarrow$ anywhere that $V(x) \propto x^2$ or approximately so.

One place it is exact is a horizontal oscillator on a frictionless surface.

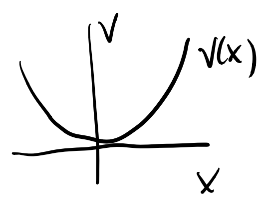


Here the force on the box is

$$\vec{F}_{spring} = -kx\,\hat{x}$$

This force is derivable from a potential

$$V(x) = \frac{1}{2}kx^2$$



So any classical system with a similar potential will have solutions similar to the SHO. In fact for many systems, we can approximate potential wells as SHO potentials.

at these local minima $V(x) \propto x^2$

This is because any Taylor series for $V(x)$ gives this,

$$V(x) \simeq V(x_0) + \left.\frac{dV}{dx}\right|_{x_0}(x-x_0) + \frac{1}{2}\left.\frac{d^2V}{dx^2}\right|_{x_0}(x-x_0)^2 + \dots$$

When x_0 corresponds to a local minimum

$$\left.\frac{dV}{dx}\right|_{x_0} = 0 \Leftarrow \text{Defines a local extrema}$$

$$V(x) \simeq V(x_0) + \frac{1}{2}\left.\frac{d^2V}{dx^2}\right|_{x_0}(x-x_0)^2 + \dots$$

We can shift zero $V(x_0)=0$ or,

$$\underbrace{V(x) - V(x_0)}_{V(x) \text{ for the purposes of } F.} \simeq \frac{1}{2}\underbrace{\left.\frac{d^2V}{dx^2}\right|_{x_0}}_{\text{always positive for local min!}}(x-x_0)^2$$

However you get there, (4)

$\boxed{V(x) = \frac{1}{2}k\,x^2}$ describes the SHO.

This model produces <u>sinusoidal solutions</u>

$$F = -kx \implies m\frac{d^2x}{dt^2} = -kx$$

$$m\ddot{x} = -kx \implies \boxed{\ddot{x} = -\frac{k}{m}x}$$

let $\omega^2 \equiv k/m$ $\qquad \ddot{x} = -\omega^2 x$

So, $\boxed{x(t) = A\cos(\omega t + \phi)}$

Classical SHO

the Hamiltonian for this 1D SHO is,

$$H = T + V = \frac{p^2}{2m} + V(x) = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

for the QHO, $k = m\omega^2$ is a better choice as there's no spring, but a just a restoring potential.

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$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$ Classical SHO Ham.

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$ Quantum Ham.