Now that we have these energy eigenstates (1)
for Hydrogen-like systems,

$$|n\ell m\rangle \doteq \psi_{n\ell m}(r,\theta,\phi) = R_{n\ell}(r)\, Y_\ell^m(\theta,\phi)$$

we can look at there time evolution
and superposition.

Time Evolution

B/c $|n\ell m\rangle$ are energy eigenstates, the
time evolution of a given state is simply,

$$|\psi(t)\rangle \doteq \psi_{n\ell m}(r,\theta,\phi,t) = R_{n\ell}(r)\, Y_\ell^m(\theta,\phi)\, e^{-iE_n t/\hbar}$$

where $E_n = -\frac{1}{2n^2}\left(\frac{Ze^2}{4\pi\varepsilon_0}\right)\frac{m_e}{\hbar^2} \quad n = 1, 2, 3, \ldots$

Superposition States

If $|\psi(t)\rangle$ is the result of
a superposition of energy eigenstates

then we time evolve each state,

$$|\psi(t)\rangle \doteq \psi(r,\theta,\phi,t)$$

$$= \sum_{n,l,m} C_{nlm} R_{nl}(r) Y_l^m(\theta,\phi) e^{-iE_n t/\hbar}$$

note E_n depends on n only so that we expect degeneracy!

We can determine coeffs by using projection (as usual)

$$C_{nlm} = \langle nlm | \psi(t=0) \rangle$$

$$= \int_0^\infty r^2 dr \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi \; R_{nl}^*(r) Y_l^{m*}(\theta,\phi) \psi(r,\theta,\phi,0)$$