The Solution we have sought thus far is,

$$Y(\theta,\phi) = \Theta(\theta)\,\Phi(\phi)$$

We have found,

$$\Theta(\theta) = (-1)^m \frac{(2\ell+1)}{2} \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(\cos\theta) \quad , m \geq 0$$

and

$$\Phi(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}$$

The appropriately normalized product gives,

$$\boxed{Y_\ell^m(\theta,\phi) = (-1)^{(m+|m|/2)} \sqrt{\frac{(2\ell+1)\,(\ell-|m|)!}{4\pi\,(\ell+|m|)!}}\, P_\ell^m(\cos\theta)\, e^{im\phi}}$$

where $\ell = 0, 1, 2, 3, \ldots$ and $m = -\ell, -\ell+1, \ldots, \ell-1, \ell$

Our sign choice is conventional and gives,

$$Y_\ell^{-m}(\theta,\phi) = (-1)^m Y_\ell^{m*}(\theta,\phi)$$

These are the <u>Spherical Harmonics</u>

and they are position representations of angular momentum eigenstates.

$$|\ell m\rangle \doteq Y_\ell^m(\theta,\phi)$$

They are tabulated online and in QM books,

l	m	Y_l^m
0	0	$Y_0^0 = \frac{1}{\sqrt{4\pi}}$
1	0	$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$
	± 1	$Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}$

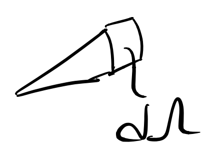
etc. . . . .

Properties of the Spherical Harmonics

① They are orthonormal on the unit sphere

$$\langle l_1 m_1 | l_2 m_2\rangle = \int_0^{2\pi}\int_0^{\pi} Y_{l_1}^{m_1 *} Y_{l_2}^{m_2} \sin\theta\, d\theta\, d\phi = \delta_{l_1 l_2}\delta_{m_1 m_2}$$

let $d\Omega = \sin\theta\, d\theta\, d\phi$ (solid angle)



$$\int Y_{l_1}^{m_1 *} Y_{l_2}^{m_2}\, d\Omega = \delta_{l_1 l_2}\delta_{m_1 m_2}$$

② they form a complete basis

given a smooth $\psi(\theta,\phi)$,

$$\psi(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} c_{lm} Y_l^m(\theta,\phi)$$

the Laplace series

Where coeffs $c_{\ell m}$ are found via projection, ③

$$\langle \ell m | \psi \rangle = c_{\ell m} = \int_0^{2\pi}\int_0^{\pi} Y_\ell^{m*}(\theta,\phi)\, \psi(\theta,\phi) \sin\theta \, d\theta \, d\phi$$

③ how Y transforms under parity $\vec{r} \to -\vec{r}$ depends on ℓ (angular momentum)

$$Y_\ell^m(\pi-\theta, \phi+\pi) = (-1)^\ell Y_\ell^m(\theta,\phi)$$

④ they are eigenstates of H, L^2, & L_z

$$H Y_\ell^m = \frac{\hbar^2}{2I} \ell(\ell+1) Y_\ell^m$$

$$L^2 Y_\ell^m = \ell(\ell+1)\hbar^2 Y_\ell^m$$

$$L_z Y_\ell^m = m\hbar \, Y_\ell^m$$

⑤ they exhibit degeneracy.

e.g. $E = \frac{\hbar^2}{2I} \ell(\ell+1)$ is $2\ell+1$ degenerate (for each m).

Thus we must sum over all degenerate states,

$$P_{E_\ell} = \sum_{m=-\ell}^{\ell} |\langle \ell m|\Psi\rangle|^2$$ prob of E_ℓ 4

or for L_z all $\ell \geq m$ are possible

$$P_{L_z = m\hbar} = \sum_{\ell=m}^{\infty} |\langle \ell m|\Psi\rangle|^2$$