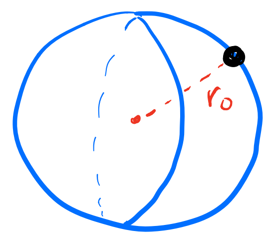
So far, we have developed the general eigenvalue eqn for a central potential & then we explored the solution in a limited case (where $r = r_0$ & $\theta = \theta_0$).

We will now continue our exploration with $r = r_0$, but θ is free. This is the "particle on a sphere"



That eigenvalue eqn is now,

$$\boxed{\frac{-\hbar^2}{2\mu r_0^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\psi + V(r_0)\,\psi = E\psi}$$

this is just the position representation of

$$H_{sphere}|E\rangle = E|E\rangle$$

As we have earlier we limit ourselves to $\psi(r_0, \theta, \phi) = Y(\theta, \phi)$

and set $V(r_0) = 0$

(2)

We also identify $\mu r_0^2 = I$ the moment of inertia for classical particle with mass μ. thus we simplify our analysis to,

$$\underbrace{-\frac{\hbar^2}{2I}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)}_{L^2 \text{ operator}} Y(\theta,\phi) = E\,Y(\theta,\phi)$$

$$\frac{L^2}{2I} Y = EY$$

We had separated our solution earlier,

$$Y(\theta,\phi) = \Theta(\theta)\,\Phi(\phi)$$

from before $L^2 Y(\theta,\phi) = A\hbar^2 Y(\theta,\phi)$

thus we expect $A = \ell(\ell+1)$ and $A = \frac{2I}{\hbar^2}E$

So E is quantized!

When we plugged in $Y = \Theta\Phi$ into our differential eqn, we obtained,

$$\left(\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) - B\frac{1}{\sin^2\theta}\right)\Theta(\theta) = -A\,\Theta(\theta)$$

$$\frac{d^2\Phi(\phi)}{d\phi^2} = -B\,\Phi(\phi)$$

Our solution to the particle on a ring gave us $B = m^2$ so that (3)

$$\boxed{\left(\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\right)\Theta(\theta) = -A\,\Theta(\theta)}$$

we must now solve this differential equ. →

(This is done in many books including McIntyre. As we do not need to derive this more than once, we will only highlight parts of that solution)

We introduce $z = \cos\theta$ and $P(z) = \Theta(\theta)$.

this gives $\sin\theta = \sqrt{1-z^2}$

Thus our boxed equ above can be rewritten as the "associated Legendre Equation",

$$\boxed{\left[(1-z^2)\frac{d^2}{dz^2} - 2z\frac{d}{dz} + A - \frac{m^2}{(1-z^2)}\right]P(z) = 0}$$

if we take the case $m=0$, we obtain "Legendre's Equation"

$$\boxed{\left((1-z^2)\frac{d^2}{dz^2} - 2z\frac{d}{dz} + A\right)P(z) = 0}$$

or,

$$\left(\frac{d^2}{dz^2} - \frac{2z}{(1-z^2)}\frac{d}{dz} + \frac{A}{(1-z^2)}\right)P(z) = 0$$

Note: there are singularities at $z = \pm 1$ or $\theta = 0, \pi$ (the poles) 4

Building a Series Solution

The approach we will take to solve Legendre's equation will use a series solution. That is we propose we can find a solution of the form,

$$P(z) = \sum_{n=0}^{\infty} a_n z^n$$

and we plug it in to find conditions on a_n.

Note that: $$\frac{dP}{dz} = \sum_{n=0}^{\infty} n a_n z^{(n-1)}$$

and $$\frac{d^2P}{dz^2} = \sum_{n=0}^{\infty} n(n-1) a_n z^{(n-2)}$$

Subbing into the last boxed eqn above yields,

$$\sum_{n=0}^{\infty} n(n-1) a_n z^{(n-2)} - z^2 \sum_{n=0}^{\infty} n(n-1) a_n z^{(n-2)}$$
$$- 2z \sum_{n=0}^{\infty} n a_n z^{(n-1)} + A \sum_{n=0}^{\infty} a_n z^n = 0$$

which gives,

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$$\boxed{\sum_{n=0}^{\infty} n(n-1)a_n z^{(n-2)} - \sum_{n=0}^{\infty} n(n-1)a_n z^n - 2\sum_{n=0}^{\infty} n a_n z^n + A\sum_{n=0}^{\infty} a_n z^n = 0}$$

Note:

$$\sum_{n=0}^{\infty} n(n-1)a_n z^{(n-2)} = \underbrace{0(-1)a_0 z^{-2}}_{\substack{n=0 \\ =0}} + \underbrace{1(0)a_1 z^{-1}}_{\substack{n=1 \\ =0}} + \underbrace{2(1)a_2 z^0}_{\substack{n=2 \\ \neq 0}} + \dots \longrightarrow$$

thus we make an index shift, $n \to n+2$

$$\sum_{n=0}^{\infty} n(n-1)a_n z^{(n-2)} = \sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2} z^n$$

the first two terms still vanish $n=-2$ & $n=-1$, so

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} z^n = \sum_{n=0}^{\infty} n(n-1)a_n z^{n-2}$$

ok so back to the boxed eqn, all terms are now z^n,

$$\underbrace{\sum_{n=0}^{\infty} \Big(a_{n+2}(n+2)(n+1) - a_n n(n-1) - 2a_n n + Aa_n n \Big) z^n = 0}$$

holds for any z & n so, the coefficients must vanish!

$$a_{n+2}(n+2)(n+1) - a_n n(n-1) - 2a_n n + Aa_n n = 0$$

So that,

$$\boxed{a_{n+2} = \frac{n(n+1) - A}{(n+2)(n+1)} a_n}$$

this recurrence relationship tells us how to get coeffs given a_0 or a_1.

Even Coeff's

$$a_2 = \frac{0(0+1) - A}{(0+2)(0+1)} a_0 = \frac{-A}{2} a_0$$

$$a_4 = \frac{2(2+1) - A}{(2+2)(2+1)} a_2 = \frac{6-A}{12} a_2 = -\frac{(6-A)(A)}{24} a_0$$

Odd Coeff's

$$a_3 = \frac{1(1+1) - A}{(1+2)(1+1)} a_1 = \frac{2-A}{6} a_1$$

$$a_5 = \frac{3(3+1) - A}{(3+2)(3+1)} a_3 = \frac{12-A}{20} a_3 = \frac{(12-A)(2-A)}{120} a_1$$

Thus our series solution is,

$$P(z) = \sum_{n=0}^{\infty} a_n z^n$$

$= a_0 z^0 + a_1 z^1 + a_2 z^2 + \dots$ where we can write everything in terms of a_0 & a_1,

$$P(z) = a_0\left(z^0 - \frac{A}{2}z^2 + \dots\right) + a_1\left(z^1 + \frac{2-A}{6}z^3 + \dots\right)$$

But we have a convergence problem if 7

$n \to \infty$ in general,

$$\lim_{n\to\infty}\left(\frac{a_{n+2}}{a_n} = \frac{n(n+1)-A}{(n+2)(n+1)}\right) \simeq 1$$

But! If we have a stronger condition on the limit of n, we might be ok.

We require so n_{max} such that the recurrence relation terminates

if $A = n_{max}(n_{max}+1)$ then,

$$a_{n_{max}+2} = \frac{0}{(n_{max}+2)(n_{max}+1)} a_{n_{max}} = 0$$

We already expect $A = \ell(\ell+1)$ given

$L^2 Y = A\hbar^2 Y$ and $L^2|\ell m\rangle = \ell(\ell+1)\hbar^2|\ell m\rangle$

So this is consistent with prior work with $\ell = 0, 1, 2, 3, \dots$

Legendre Polynomials

The special values of $A = \ell(\ell+1)$ give rise to polynomials of degree ℓ, $P_\ell(z)$ the "Legendre Polynomials"

We can calculate them via,

$$\boxed{P_\ell(z) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dz^\ell} (z^2-1)^\ell \quad \text{Rodriguez Formula}}$$

$P_0(z) = 1$

$P_1(z) = z$

$P_2(z) = \frac{1}{2}(3z^2-1)$

$P_3(z) = \frac{1}{2}(5z^3-3z)$

$P_4(z) = \frac{1}{8}(35z^4-30z^2+3)$

etc.

Legendre Polynomials are orthogonal!

$$\boxed{\int_{-1}^{1} P_k^*(z)\, P_\ell(z)\, dz = \frac{2}{2\ell+1} \delta_{k\ell}}$$

Now that we know $A = \ell(\ell+1)$ we can explore cases where $m \neq 0$

Going back to our original diff. E.Q. for $P(z)$,

$$\boxed{\left((1-z^2)\frac{d^2}{dz^2} - 2z\frac{d}{dz} + \ell(\ell+1) - \frac{m^2}{(1-z^2)} \right) P(z) = 0}$$

This differential eqn is well-studied and its solutions are the "associated Legendre functions."

$$\boxed{\begin{aligned} P_\ell^{m}(z) = P_\ell^{-m}(z) &= (1-z^2)^{m/2} \frac{d^m}{dz^m} P_\ell(z) \\ &= \frac{1}{2^\ell \ell!} (1-z^2)^{m/2} \frac{d^{m+\ell}}{dz^{m+\ell}} (z^2-1)^\ell \end{aligned}}$$

Note: $\dfrac{d^{m+\ell}}{dz^{m+\ell}} (z^2+1)^\ell$ takes the $(m+\ell)$ derivative of a polynomial of order ℓ

if $m>\ell$ then $\dfrac{d^{m+\ell}}{dz^{m+\ell}} (z^2+1)^\ell = 0$

so

$m = -\ell, -\ell+1, \ldots, 0, \ldots, \ell-1, \ell$

that is $|m| \leq \ell$ integers only

These associated Legendre polynomials are orthogonal:

$$\int_{-1}^{1} P_\ell^m(z) P_q^m(z)\, dz = \frac{2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!} \delta_{\ell q}$$

Our Θ solution

Originally we wrote $z = \cos\theta$ so the Θ eigenstates determined from $P_\ell^m(z)$ are,

$$\boxed{\Theta_\ell^m(\theta) = (-1)^m \frac{(2\ell+1)}{2} \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(\cos\theta) \quad , \quad m \geq 0}$$

and

$$\Theta_\ell^{-m}(\theta) = (-1)^m \Theta_\ell^m(\theta) \quad , \quad m \geq 0$$

with the orthogonality relationship,

$$\boxed{\int_0^\pi \Theta_\ell^m(\theta)\, \Theta_q^m(\theta) \sin\theta\, d\theta = \delta_{\ell q}}$$

and $P_\ell^m(\cos\theta)$ is,

$P_0^0 = 1$ $\qquad$ $P_2^0 = \frac{1}{2}(3\cos^2\theta - 1)$

$P_1^0 = \cos\theta$ $\qquad$ $P_2^1 = 3\sin\theta\cos\theta$

$P_1^1 = \sin\theta$ $\qquad$ $P_2^2 = 3\sin^2\theta$ $\qquad$ etc.