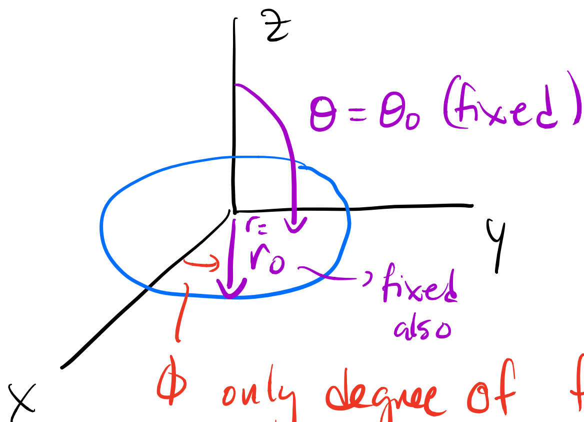
We can begin our formal exploration of our separable solutions by considering a particle bound to a ring.



ϕ only degree of freedom (1D problem!)

B/c $\theta = \theta_0$ & $r = r_0$,

$$-\frac{\hbar^2}{2\mu}\frac{1}{r_0^2}\frac{d^2\psi}{d\phi^2} + V(r_0)\psi = E\psi$$

We can set $V(r_0) = 0$ for this setup ∴

$\psi(r,\theta,\phi) = \Phi(\phi)$ only 1D!

$$-\frac{\hbar^2}{2\mu r_0^2}\frac{d^2\Phi}{d\phi^2} = E\Phi$$

Note that $\mu r_0^2 = I$ moment of inertia of particle

$$\boxed{-\frac{\hbar^2}{2I}\frac{d^2\Phi}{d\phi^2} = E\Phi}$$

In our separable solution, we found, (2)

$$\frac{d^2\Phi}{d\phi^2} = -B\Phi \quad \text{so } B = \frac{2I}{\hbar^2}E$$

In position space, $L_z \doteq -i\hbar \frac{d}{d\phi}$ so that,

$$L_z^2 = -\hbar^2 \frac{d^2}{d\phi^2}$$

And thus,

$$\frac{L_z^2}{2I}\Phi = E\Phi$$

this tracks with H_{sys}, $H_{sys} = T = L_z^2/2I$ for this system.

Thus, eigenstates of L_z are also energy eigenstates ($V(r_0) = 0$)

$$L_z|l\, m_l\rangle = m_l\hbar|l\, m_l\rangle$$

$$L_z^2|l\, m_l\rangle = m_l^2\hbar^2|l\, m_l\rangle$$

Thus,

$$\boxed{H_{sys}|l\, m_l\rangle = \frac{L_z^2}{2I}|l\, m_l\rangle = \frac{m_l^2\hbar^2}{2I}|l\, m_l\rangle}$$

or $\boxed{E = \frac{m_\ell^2 \hbar^2}{2I} \quad \text{where } I = \mu r_0^2}$

This is great! We found eigenstates of H & the associated energies!

But, what about the position representation?

Position Rep.

$$\frac{d^2\Phi}{d\phi^2} = -B\Phi \Rightarrow \Phi(\phi) = Ne^{\pm i\sqrt{B}\phi}$$

general solution!

The ring problem requires $\Phi(\phi) = \Phi(\phi + 2\pi)$

$\Rightarrow$ Φ must be single valued.

So $\sqrt{B}$ must be real (periodic solution)

Finally, to match the periodicity $\sqrt{B}$ must be an integer,

$$\sqrt{B} = m = 0, \pm 1, \pm 2, \ldots$$

So,

$$\boxed{\Phi(\phi) = N e^{im\phi} \qquad m = 0, \pm 1, \pm 2, \ldots}$$

Let's check expectations,

$$L_z | l\, m \rangle = m\hbar\, | l m \rangle$$

$L_z \doteq -i\hbar \frac{d}{d\phi}$ so that

$$L_z \Phi(\phi) = -i\hbar \frac{d}{d\phi}\left(N e^{im\phi}\right)$$

$$= -i\hbar\,(im)\, N e^{im\phi} = m\hbar\, N e^{im\phi}$$

$$\boxed{L_z \Phi(\phi) = m\hbar\, \Phi(\phi) \quad \checkmark}$$

Let's normalize $\Phi(\phi)$,

$$\langle \Phi | \Phi \rangle = 1$$

$$\doteq \int_0^{2\pi} \Phi^* \Phi \, d\phi = |N|^2 \int_0^{2\pi} d\phi = 2\pi |N|^2$$

5

$$\boxed{N = \frac{1}{\sqrt{2\pi}} \quad \therefore \quad \Phi(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}}$$

These states are orthogonal,

$$\langle k | m \rangle = \delta_{km} = \int_0^{2\pi} \Phi_k(\phi)\, \Phi_m(\phi)\, d\phi$$