(1)

We can rewrite the PDE that characterized our central potential eigenvalue problem by defining the L^2 operator in position space

$$L^2 \doteq -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

this gives us,

$$\frac{-\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{1}{\hbar^2 r^2}L^2\right]\psi(r,\theta,\phi)$$

$$+V(r)\psi(r,\theta,\phi)=E\psi(r,\theta,\phi)$$

This is still a 3D PDE that appears quite complex (in general, it is!). We can make headway by considering a particular kind of solution:

$$\boxed{\psi(r,\theta,\phi)=R(r)Y(\theta,\phi)}$$

proposed separable solution

We can propose this form of the solution and see what happens. Note R is a function of r only and Y a function of θ,ϕ only!

2

$$\frac{-\hbar^2}{2\mu}\left[Y(\theta,\phi)\,\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)R(r) - \frac{1}{\hbar^2 r^2}R(r)\,L^2 Y(\theta,\phi)\right]$$

$$+\,V(r)\,R(r)\,Y(\theta,\phi) = E\,R(r)\,Y(\theta,\phi)$$

After plugging in we divide by ψ,

Become regular derivatives.

$$\frac{-\hbar^2}{2\mu}\left[\frac{1}{R}\,\frac{1}{r^2}\,\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{1}{Y}\,\frac{1}{\hbar^2 r^2}\,L^2 Y\right] + V(r) = E$$

The separation of variables "game" is to produce functions of the separated variables (i.e. r and θ,ϕ)

Multiply by r^2 throughout.

$$\frac{-\hbar^2}{2\mu}\left[\frac{1}{R}\,\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{1}{Y}\,\frac{1}{\hbar^2}L^2 Y\right] + V(r)\,r^2 = E r^2$$

or,

$$\underbrace{\frac{1}{R}\left(\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)\right) - \frac{2\mu}{\hbar^2}\left(E-V(r)\right)r^2}_{\text{function of } r \text{ only}} = \underbrace{\frac{1}{\hbar^2}\,\frac{1}{Y}\,L^2 Y}_{\text{function of } \theta,\phi \text{ only.}}$$

3

Key Point: this equality hold regardless of r or θ & ϕ. so both sides must be equal to a constant. Call it 'A'.

$$\boxed{\begin{aligned} &① \quad L^2 Y(\theta,\phi) = A\hbar^2 Y(\theta,\phi) \\ &② \left[-\frac{\hbar^2}{2\mu r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) + V(r) + A\frac{\hbar^2}{2\mu r^2}\right]R(r) = E R(r) \end{aligned}}$$

We will deal w/ eqn. 2 later because we need a particular $V(r)$. But Eq. 1 doesn't require a $V(r)$ so let's explore it.

$$L^2 Y(\theta,\phi) = A\hbar^2 Y(\theta,\phi)$$

Let's try to separate variables again,

$$\boxed{Y(\theta,\phi) = \Theta(\theta)\,\Phi(\phi)}$$

we know from $L^2 |l\, m_l\rangle = l(l+1)\hbar^2 |l\, m_l\rangle$ that we expect $A = l(l+1)$

(But we do this later)

With,

$$L^2 \doteq -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

4

$$L^2 Y(\theta,\phi) = A\hbar^2 Y(\theta,\phi)$$

$$-\cancel{\hbar^2}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{d\phi^2}\right]\Theta(\theta)\Phi(\phi)$$

$$= A\cancel{\hbar^2}\,\Theta(\theta)\Phi(\phi)$$

$$-\Phi\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\Theta\frac{1}{\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = A\Theta\Phi$$

Divide by $Y(\theta,\phi)$,

$$-\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\Phi}\frac{1}{\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = A$$

Multiply by $\sin^2\theta$

$$-\frac{1}{\Theta}\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = A\sin^2\theta$$

$$\underbrace{A\sin^2\theta+\frac{1}{\Theta}\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)}_{\theta\text{ only!}} = \underbrace{\frac{-1}{\Phi}\frac{d^2\Phi}{d\phi^2}}_{\phi\text{ only!}}$$

5

Set equal to constant, B.

$$\boxed{\begin{aligned} &(3)\quad \frac{d^2\Phi(\phi)}{d\phi^2} = -B\,\Phi(\phi) \\ &(4)\quad \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta(\theta)}{d\theta}\right) - \frac{B}{\sin^2\theta}\Theta(\theta) = -A\,\Theta(\theta) \end{aligned}}$$

We will explore implications of eqns. 3 & 4 soon. We start with 3.