Scattering

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So far we have focused on bound states where $E < V_0$ or on a particle that is free ($V = 0$ everywhere).

But we have built up the mathematical architecture to deal with situations where $E > V_0$ but $V_0 \neq 0$.

These are called scattering states and are common in many experimental situations in atomic, nuclear, & high energy physics!

Best to start by example with a finite square well but with $E > V_0$.

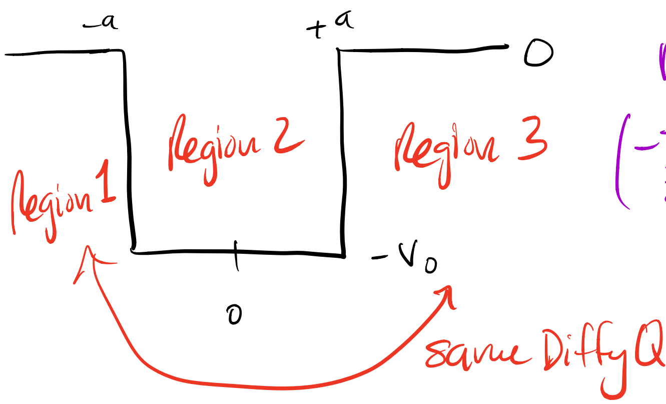
Example: Scattering off a finite Well

Let's assume,

$$V(x) = \begin{cases} 0 & x < -a \\ -V_0 & -a < x < a \\ 0 & x > a \end{cases}$$

Here we've set the zero of energy so that $E > V_0$ means $E > 0$ always. This is useful because it ensures the sign of E is always positive!

We get the following,



Region 2 $|x| < a$

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - V_0\right)\varphi_E(x) = E\varphi_E(x)$$

Region 1+3 $|x| > a$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\varphi_E(x) = E\varphi_E(x)$$

<u>Region 2</u>

$$\frac{d^2}{dx^2}\varphi_E(x) = -\frac{2m}{\hbar^2}(E+V_0)\varphi_E(x) = -k_2^2\varphi_E(x)$$

$$k_2^2 \equiv \frac{2m}{\hbar^2}(E+V_0) \qquad k_2 > 0 \text{ always!}$$

<u>Region 1 + 3</u>

$$\frac{d^2}{dx^2}\varphi_E(x) = -\frac{2m}{\hbar^2}E\varphi_E(x) = -k_1^2\varphi_E(x)$$

$$k_1^2 \equiv \frac{2m}{\hbar^2}E \qquad k_1 > 0 \text{ always!}$$

Thus our solutions a sinusoidal everywhere!

$$\varphi_E(x) = \begin{cases} Ae^{ik_1x} + Be^{-ik_1x} & x < -a \\ Ce^{ik_2x} + De^{-ik_2x} & -a < x < a \\ Fe^{ik_1x} + Ge^{-ik_1x} & x > a \end{cases}$$

Pause: this looks very familiar. Typically, we now proceed with matching Boundary Conditions and using normalization to find our particular solutions.

However we have 7 unknowns! (A, B, C, D, F, G & E the energy!)

Fortunately this general problem is not the setup we often have experimentally. Indeed it is far more typical we send in particles of some known Energy to interact with other particles. 4

the well is a model for the interaction.

So typically,

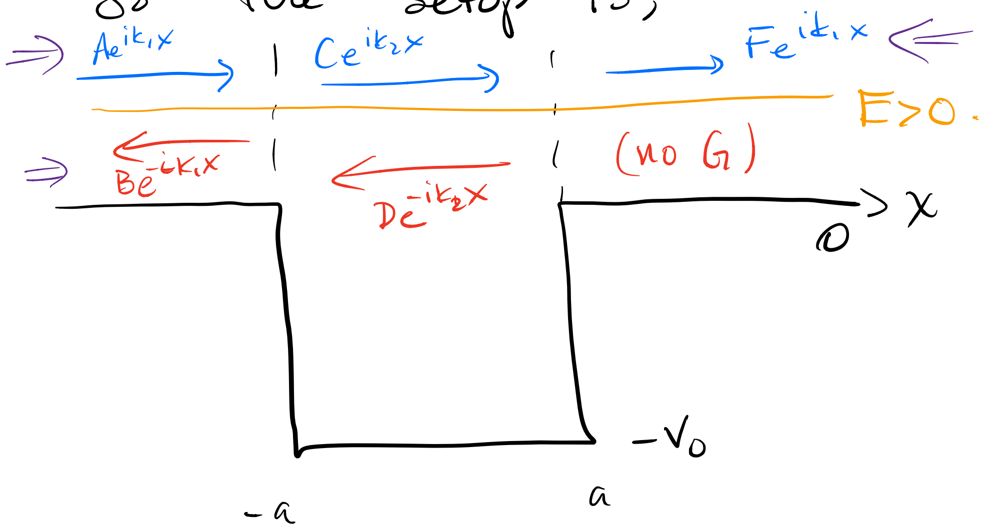
(1) we set A incoming amplitude of particles (proxy for luminosity of the beam)

(2) we set the energy of the particles, E

&

(3) no particles are directed backwards in Region 3.

So the setup is,



* Finally, because the details of the well are not typically of experimental interest, we often choose to solve for A, B, & F to see the resulting reflection (B) and transmission (F). Because we control A, we typically solve for B/A & F/A. 5

So we will start with,

$$\varphi_E(x) = \begin{cases} Ae^{ik_1x} + Be^{-ik_1x} & x < -a \\ Ce^{ik_2x} + De^{-ik_2x} & -a < x < a \\ Fe^{+ik_1x} & x > a \end{cases}$$

B/c the well is finite we expect,

$\varphi(a)$ and $\varphi(-a)$ as well as $\left.\frac{d\varphi}{dx}\right|_{\pm a}$ to be continuous.

→ And given the choice we elimante C & D for B & F.

$\varphi_E(-a)$: $Ae^{-ik_1a} + Be^{+ik_1a} = Ce^{-ik_2a} + De^{+ik_2a}$

$\frac{d\varphi_E}{dx}(-a)$: $ik_1Ae^{-ik_1a} - ik_1Be^{+ik_1a} = ik_2Ce^{-ik_2a} - ik_2De^{+ik_2a}$

$\varphi_E(+a)$: $Ce^{ik_2a} + De^{-ik_2a} = Fe^{ik_1a}$

$\frac{d\varphi_E}{dx}(+a)$: $ik_2Ce^{ik_2a} - ik_2De^{-ik_2a} = ik_1Fe^{ik_1a}$

eliminate C & D for F using last two eqns.

$$Ce^{ik_2a} = Fe^{ik_1a} - De^{-ik_2a}$$

$$ik_2\left(Fe^{ik_1a} - De^{-ik_2a}\right) - ik_2De^{-ik_2a} = ik_1Fe^{ik_1a}$$

$$-2k_2De^{-ik_2a} = (k_1-k_2)Fe^{ik_1a}$$

$$\boxed{D = \left(\frac{k_2-k_1}{2k_2}\right)e^{i(k_1+k_2)a}F}$$

thus,

$$Ce^{ik_2a} = Fe^{ik_1a} - \frac{(k_2-k_1)}{2k_2}e^{i(k_1+k_2)a}Fe^{-ik_2a}$$

$$Ce^{ik_2a} = Fe^{ik_1a} + \frac{(k_1-k_2)}{2k_2}e^{ik_1a}F$$

$$Ce^{ik_2a} = e^{ik_1a}F\left(1+\frac{k_1-k_2}{2k_2}\right)$$

$$Ce^{ik_2a} = e^{ik_1a}F\left(\frac{2k_2+k_1-k_2}{2k_2}\right)$$

$$Ce^{ik_2a} = e^{ik_1a}F\left(\frac{k_2+k_1}{2k_2}\right)$$

$$\boxed{C = \left(\frac{k_2+k_1}{2k_2}\right)e^{i(k_1-k_2)a}F}$$

Now we use the first two eqns.

$$Ae^{-ik_1a} + Be^{ik_1a} = Ce^{-ik_2a} + De^{+ik_2a}$$

$$ik_1Ae^{-ik_1a} - ik_1Be^{ik_1a} = ik_2Ce^{-ik_2a} - ik_2De^{+ik_2a}$$

$$Ae^{-ik_1a} + Be^{ik_1a} = \left(\frac{k_2+k_1}{2k_2}\right)e^{i(k_1-k_2)a}e^{-ik_2a}F$$

$$+\left(\frac{k_2-k_1}{2k_2}\right)e^{i(k_1+k_2)}e^{+ik_2a}F$$

$$\boxed{Ae^{-ik_1a} + Be^{ik_1a} = \left[\left(\frac{k_2+k_1}{2k_2}\right)e^{i(k_1-2k_2)a} + \left(\frac{k_2-k_1}{2k_2}\right)e^{i(k_1+2k_2)a}\right]F}$$

$$k_1 A e^{-ik_1 a} - k_1 B e^{ik_1 a} = k_2\left(\frac{k_2+k_1}{2k_2}\right) e^{i(k_1-k_2)a} e^{-ik_2 a} F$$

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$$- k_2\left(\frac{k_2-k_1}{2k_2}\right) e^{i(k_1+k_2)a} e^{ik_2 a} F$$

$$\boxed{A e^{-ik_1 a} - B e^{ik_1 a} = \left(\left(\frac{k_2+k_1}{2k_1}\right) e^{i(k_1-2k_2)a} - \frac{(k_2-k_1)}{2k_1} e^{i(k_1+2k_2)a}\right) F}$$

$$2A e^{-ik_1 a} = \left[\left(\frac{k_2+k_1}{2k_2}\right) e^{i(k_1-2k_2)a} + \left(\frac{k_2-k_1}{2k_2}\right) e^{i(k_1+2k_2)a}\right.$$

$$\left. + \left(\frac{k_2+k_1}{2k_1}\right) e^{i(k_1-2k_2)a} - \left(\frac{k_2-k_1}{2k_1}\right) e^{i(k_1+2k_2)a}\right] F$$

$$2A = \left[\left(\frac{k_2+k_1}{2k_2}\right) e^{-i2k_2 a} + \left(\frac{k_2-k_1}{2k_2}\right) e^{+i2k_2 a}\right.$$

$$\left. + \left(\frac{k_2+k_1}{2k_1}\right) e^{-i2k_2 a} - \left(\frac{k_2-k_1}{2k_1}\right) e^{i2k_2 a}\right] F$$

$$2A = \left[\left(\frac{k_2+k_1}{2k_2} + \frac{k_2+k_1}{2k_1}\right) e^{-i2k_2 a} + \left(\frac{k_2-k_1}{2k_2} - \frac{k_2-k_1}{2k_1}\right) e^{i2k_2 a}\right) F$$

$$= \left(\left(\frac{k_2 k_1 + k_1^2 + k_2^2 + k_1 k_2}{2k_1 k_2}\right) e^{-i2k_2 a} + \left(\frac{k_2 k_1 - k_1^2 - k_2^2 + k_2 k_1}{2k_2 k_1}\right) e^{+i2k_2 a}\right) F$$

$$= \left(\frac{(k_1+k_2)^2}{2k_1k_2} e^{-i2k_2a} - \frac{(k_1-k_2)^2}{2k_1k_2} e^{+i2k_2a} \right) F$$ 9

$$A = \frac{F}{4k_1k_2}\left((k_1+k_2)^2 e^{-i2k_2a} - (k_1-k_2)^2 e^{+i2k_2a} \right)$$

$$= \frac{F}{4k_1k_2}\Big[k_1^2\left(e^{-i2k_2a} - e^{+i2k_2a}\right) + 2k_1k_2\left(e^{-i2k_2a} + e^{+i2k_2a}\right)$$
$$k_2^2\left(e^{-i2k_2a} - e^{+i2k_2a}\right)\Big]$$

$$= \frac{F}{2k_1k_2}\Big[-ik_1^2\left(\frac{e^{i2k_2a} - e^{-i2k_2a}}{2i} \right)$$
$$+ 2k_1k_2\left(\frac{e^{+i2k_2a} + e^{-i2k_2a}}{2} \right)$$
$$-ik_2^2 \left(\frac{e^{+ik_2a} - e^{-ik_2a}}{2i} \right)\Big]$$

$$= \frac{F}{2k_1k_2}\Big[-ik_1^2 \sin(2k_2a) + 2k_1k_2\cos(2k_2a)$$
$$-ik_2^2 \sin(2k_2a)\Big]$$

$$= F\left(\cos(2k_2a) - i\,\frac{k_1^2+k_2^2}{2k_1k_2}\sin(2k_2a)\right)$$

$$\boxed{\frac{F}{A} = \frac{1}{\cos(2k_2 a) - i\left(\frac{k_1^2+k_2^2}{2k_1k_2}\right)\sin(2k_2a)}}$$ (10)

and now B/A,

$$2Be^{+ik_1a} = \left[\left(\frac{k_2+k_1}{2k_2}\right)e^{i(k_1-2k_2)a} + \left(\frac{k_2-k_1}{2k_2}\right)e^{i(k_1+2k_2)a}\right]F$$

$$-\left(\left(\frac{k_2+k_1}{2k_1}\right)e^{i(k_1-2k_2)a} - \frac{(k_2-k_1)}{2k_1}e^{i(k_1+2k_2)a}\right)F$$

$$2B = F\left(\frac{k_2+k_1}{2k_2}e^{-i2k_2a} + \frac{k_2-k_1}{2k_2}e^{+i2k_2a} - \frac{(k_2+k_1)}{2k_1}e^{-i2k_2a} + \frac{(k_2-k_1)}{2k_1}e^{i2k_2a}\right)$$

$$2B = F\left(\frac{k_1^2+k_2k_1-k_1k_2-k_2^2}{2k_2k_1}e^{-i2k_2a} + \frac{k_2k_1-k_1^2-k_2k_1+k_2^2}{2k_2k_1}e^{+i2k_2a}\right)$$

$$2B = \frac{F}{2k_1k_2}\left(-k_1^2\left(e^{i2k_2a}-e^{-i2k_2a}\right)+k_2^2\left(e^{i2k_2a}-e^{-i2k_2a}\right)\right)$$

$$2B = \frac{F}{2k_1k_2}\left(-2ik_1^2 \sin(2k_2a) + 2ik_2^2 \sin(2k_2a)\right)$$

$$B = \frac{F}{2k_1k_2}\left(i(k_2^2-k_1^2)\sin(2k_2a)\right)$$

$$\boxed{\frac{B}{A} = i\,\frac{F}{A}\,\frac{k_2^2-k_1^2}{2k_1k_2}\sin(2k_2a)}$$

Reflection & Transmission

Our chief interest is on the reflection and transmission of the incoming wave. We can determine that by finding the absolute square of these ratios.

$$T = \frac{|F|^2}{|A|^2} \quad \text{and} \quad R = \frac{|B|^2}{|A|^2}$$

also $T + R = 1$

$$\frac{F}{A} = \frac{1}{\cos(2k_2 a) - i\left(\frac{k_1^2 + k_2^2}{2k_1 k_2}\right)\sin(2k_2 a)}$$

$$\frac{|F|^2}{|A|^2} = \left(\frac{1}{\cos(2k_2 a) - i\left(\frac{k_1^2 + k_2^2}{2k\ k_2}\right)\sin(2k_2 a)}\right)$$

$$\times \left(\frac{1}{\cos(2k_2 a) + i\left(\frac{k_1^2 + k_2^2}{2k_1\ k_2}\right)\sin(2k_2 a)}\right)$$

$$= \frac{1}{\cos^2(2k_2 a) + \left(\frac{k_1^2 + k_2^2}{2k_1 k_2}\right)^2 \sin^2(2k_2 a)}$$

$$= \frac{1}{1 - \sin^2(2k_2 a) + \left(\frac{k_1 + k_2}{2k_1 k_2}\right)^2 \sin^2(2k_2 a)}$$

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$$= \frac{1}{1 + \left(\left(\frac{k_1^2 + k_2^2}{2k_1k_2} \right)^2 - 1 \right) \sin^2(2k_2 a)}$$

$$= \frac{1}{1 + \left(\frac{k_1^4 + k_2^4 + 2k_1^2 k_2^2 - 4k_1^2 k_2^2}{4k_1^2 k_2^2} \right) \sin^2(2k_2 a)}$$

$$= \frac{1}{1 + \left(\frac{k_1^4 - 2k_1^2 k_2^2 + k_2^4}{4k_1^2 k_2^2} \right) \sin^2(2k_2 a)}$$

$$\boxed{T = \frac{|F|^2}{|A|^2} = \frac{1}{1 + \left(\frac{k_1^2 - k_2^2}{2k_1k_2} \right)^2 \sin^2(2k_2 a)}}$$

$$\frac{B}{A} = i \frac{F}{A} \frac{k_2^2 - k_1^2}{2k_1k_2} \sin(2k_2 a)$$

$$R = \frac{|B|^2}{|A|^2} = \frac{|F|^2}{|A|^2} \left(\frac{k_2^2 - k_1^2}{2k_1k_2} \right)^2 \sin^2(2k_2 a)$$

$$\boxed{R = \left(\frac{k_2^2 - k_1^2}{2k_1k_2}\right)^2 \sin^2(2k_2a) \Big/ \left(1 + \left(\frac{k_1^2 - k_2^2}{2k_1k_2}\right)^2 \sin^2(2k_2a)\right)}$$

Yuck! But we can use these for a given choice of E to predict the relative transmission and reflection of a beam of QM particles.