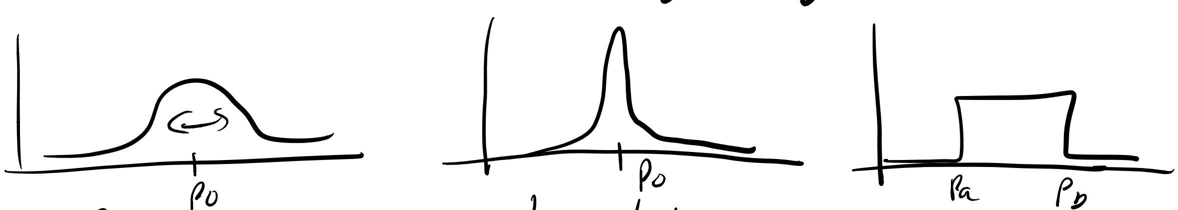
Now that we have seen how we can begin to construct wave packets using a 3 eigenstate wavefunction, we will generalize to a distribution of eigenstates given by $\phi(p)$.

Here $\phi(p)$ could be anything,



Gaussian Lorentzian uniform

etc... In these notes we will focus on Gaussian because: (a) they are common in experiments and (b) they mathematical tractable.

We start by writing $\Psi(x,0)$ with the knowledge that $\phi(p)$ is defined from $-\infty$ to $+\infty$:

$$\Psi(x,0) = \int_{-\infty}^{\infty} \underbrace{\phi(p)}_{\text{coeff for a given } p.} \psi_p(x)\, dp \quad \Big\} \text{ We are simply adding up all the momentum eigenstates.}$$

$\varphi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$ per usual so, 2

$$\psi(x,0) = \int_{-\infty}^{\infty} \phi(p) \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} dp$$

Given that the momentum eigenstates are also energy eigenstates for a free particle with $E_p = p^2/2m$, time evolution of ψ is quite simple.

$$\psi(x,t) = \int_{-\infty}^{\infty} \phi(p) \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} e^{-iE_pt/\hbar} dp$$

$$\psi(x,t) = \int_{-\infty}^{\infty} \phi(p) \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} e^{-i\frac{p^2t}{2m\hbar}} dp$$

or

$$\boxed{\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(p) e^{ip(x - \frac{p}{2m}t)/\hbar} dp}$$

time evolution of free particle general state
* we need a $\phi(p)$ to solve

This eqn might look sort of familiar. Earlier, (3)
we produced

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(p)\, e^{ipx/\hbar}\, dp$$

which is the fourier transform of $\phi(p)$.

What we have constructed is the time dependent Fourier transform of $\phi(p)$,

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(p)\, e^{ip(x - \frac{p}{2m}t)/\hbar}\, dp$$

Given that the inverse transform from earlier was,

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x)\, e^{-ipx/\hbar}\, dx$$

We expect the time dependent inverse transform to be,

$$\boxed{\phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x,t)\, e^{-ip(x - \frac{p}{2m}t)/\hbar}\, dx}$$

Example: Gaussian Distributed $\phi(p)$ 4

Let's assume we have a Gaussian $\phi(p)$ that is peaked at p_0 and has a width that is charaterized by β.

A properly normalized momentum space wavefunction, $\phi(p)$, with these attributes is given by,

$$\boxed{\phi(p) = \left(\frac{1}{2\pi\beta^2}\right)^{1/4} e^{-(p-p_0)^2/4\beta^2}}$$

The probability distribution for this wave function is simply absolute square,

$$P(p) = |\phi(p)|^2 = \frac{1}{\beta\sqrt{2\pi}}\, e^{-(p-p_0)^2/2\beta^2}$$

A typical Gaussian is given by $f(z) = \dfrac{e^{-(z-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}$

So we can read off $\mu = \langle p \rangle = p_0$

and $\sigma = \Delta p = \beta$

Ok Let's get to calculating, we want to take the time dependent Fourier transform of $\phi(p)$, 5

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(p)\, e^{+ip(x - \frac{p}{2m}t)/\hbar}\, dp$$

or

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \left(\frac{1}{2\pi\beta^2}\right)^{1/4} e^{-(p-p_0)^2/2\beta^2} e^{ipx/\hbar} e^{-i\frac{p^2}{2m}t/\hbar}\, dp$$

Yikes!

Fortunately Gaussian Integrals are "well known", aka compiled online & stuff,

$$\int_{-\infty}^{+\infty} e^{-a^2x^2+bx}\, dx = \frac{\sqrt{\pi}}{a} e^{b^2/4a^2}$$

Given this the first of a number of such complex integrals, lets unpack it. We have a polynomial form $-a^2x^2+bx$ that we seek. So lets combine all

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the exponentials above,

$$e^{blah\,1} e^{blah\,2} e^{blah\,3} = e^{(blah\,1 + blah\,2 + blah\,3)}$$

that would give,

$$\frac{-(p-p_0)^2}{2\beta^2} + \frac{ipx}{\hbar} - \frac{ip^2 t}{2m\hbar}$$

let's expand and collect p^2 & p terms,

$$-\frac{(p^2 - 2pp_0 + p_0^2)}{2\beta^2} + \frac{ix}{\hbar} p - \frac{it}{2m\hbar} p^2$$

$$= -\left(\frac{it}{2m\hbar} + \frac{1}{2\beta^2}\right)p^2 + \left(\frac{p_0}{\beta^2} + \frac{ix}{\hbar}\right)p - \frac{p_0^2}{2\beta^2}$$

Notice that these exponents are of the form $-ax^2 + bx + c$

if we exponentiate we get,

$$e^{-ax^2+bx+c} = \underbrace{e^{c}}_{\text{Const term!}} e^{-ax^2+bx}$$

p_0 is known!

$$c = -p_0^2/2\beta^2$$

So with $a = \left(\frac{it}{2m\hbar} + \frac{1}{2\beta^2}\right)$ and $b = \left(\frac{p_0}{\beta^2} + \frac{ix}{\hbar}\right)$

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then,

$$\int_{-\infty}^{\infty} e^{-ap^2+bp^2}\, dp = \sqrt{\frac{\pi}{a}}\, e^{b^2/4a^2}$$

Let's go all the way back,

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \left(\frac{1}{2\pi\beta^2}\right)^{1/4} e^{-(p-p_0)^2/2\beta^2} e^{ipx/\hbar} e^{-i\frac{p^2}{2m}t/\hbar}\, dp$$

We rewrite as,

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{1}{2\pi\beta^2}\right)^{1/4} \int_{-\infty}^{\infty} e^{c} e^{-ap^2+bp}\, dp$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{1}{2\pi\beta^2}\right)^{1/4} e^{c} \underbrace{\int_{-\infty}^{\infty} a^{-ap^2+bp}\, dp}_{\sqrt{\frac{\pi}{a}}\, e^{b^2/4a^2}}$$

$$\boxed{\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{1}{2\pi\beta^2}\right)^{1/4} e^{c} \sqrt{\frac{\pi}{a}}\, e^{b^2/4a^2}}$$

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Plug everything back in!

$$a = \left(\frac{it}{2m\hbar} + \frac{1}{2\beta^2}\right)$$

$$b = \left(\frac{p_0}{\beta^2} + \frac{ix}{\hbar}\right)$$

$$c = -p_0^2/2\beta^2$$