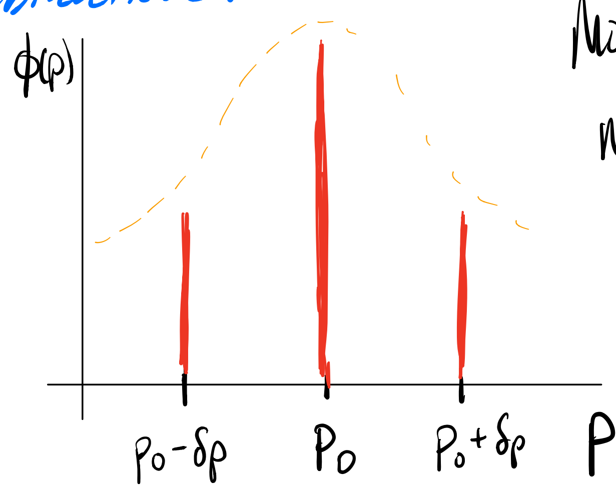
-To build up our understanding of wave packets, we will start with a discrete combination of momentum eigenstates. 1

- We will be able to see how this combination of states produces a wave packet in which a carrier wave is contained by an envelope.

Let's start with a 3 state linear combination.



Model of a Gaussian momentum distribution w/ $\frac{1}{2}$ $p_0 \pm \delta p$ and 1 p_0

As usual,

$$\psi(x,0) = \sum_j c_j \phi_{p_j}(x)$$

For normalized momentum eigenstates, 2

$$\psi(x,0) = \sum_j c_j \frac{1}{\sqrt{2\pi\hbar}} e^{ip_j x/\hbar}$$

so that

$$\boxed{\psi(x,0) = \frac{1}{\sqrt{2\pi\hbar}}\left[\frac{1}{2}e^{i(p_0-\delta p)x/\hbar} + e^{ip_0 x/\hbar} + \frac{1}{2}e^{i(p_0+\delta p)x/\hbar}\right]}$$

Now, for the free particle, momentum eigenstates are also energy eigenstates with energy $\underline{E_j = p_j^2/2m}$

So time evolution is quite straight-forward,

$$\psi(x,t) = \sum_j c_j \varphi_{p_j}(x)\, e^{-iE_j t/\hbar}$$

For p_0, $$E_{p_0} = p_0^2/2m$$

For the additional energies,

$$E_{p_0 \pm \delta p} = (p_0 \pm \delta p)^2/2m$$

3

For the sake of our sanity, assume $\delta p \ll p_0$ (the momentum distribution is fairly narrow)

$$E_{p_0 \pm \delta p} = \frac{(p_0 \pm \delta p)^2}{2m} = \frac{p_0^2 \pm 2p_0\delta p + (\delta p)^2}{2m}$$

linearize

$$\simeq \frac{p_0^2 \pm 2p_0\delta p}{2m} = \frac{p_0}{2m}\left(p_0 \pm 2\delta p\right)$$

with

$$E_0 = \frac{p_0^2}{2m} \quad \text{and} \quad E_{p_0 \pm \delta p} \simeq \frac{p_0}{2m}(p_0 \pm 2\delta p)$$

we can construct our time-evolved state,

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\left[\frac{1}{2} e^{i(p_0 - \delta p)x/\hbar} e^{-i(p_0^2 - 2p_0\delta p)t/2m\hbar} + e^{ip_0x/\hbar} e^{-ip_0^2 t/2m\hbar} + \frac{1}{2} e^{i(p_0+\delta p)x/\hbar} e^{-i(p_0^2 + 2p_0\delta p)t/2m\hbar}\right]$$

Let's factor out $e^{ip_0x/\hbar} e^{-ip_0^2 t/2m\hbar}$, 4

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\, e^{ip_0 x/\hbar}\, e^{-ip_0^2 t/2m\hbar}$$

$$\times\left[1 + \frac{1}{2} e^{-i\delta p\, x/\hbar}\, e^{ip_0 \delta p\, t/\hbar m} + \frac{1}{2} e^{+i\delta p\, x/\hbar}\, e^{-ip_0 \delta p\, t/\hbar m}\right]$$

$$\frac{e^{iy} + e^{-iy}}{2} = \cos(y) \quad \text{here } y = \left(\frac{\delta p\, x}{\hbar} - \frac{p_0 \delta p\, t}{m\hbar}\right)$$

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\, e^{ip_0 x/\hbar}\, e^{-ip_0^2 t/2m\hbar}\left[1 + \cos\left(\frac{\delta p}{\hbar} x - \frac{p_0 \delta p}{m\hbar} t\right)\right]$$

There's not much more to simplify except to maintain a $f(x-vt)$ form,

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\, e^{ip_0\underbrace{(x - \frac{p_0}{2m}t)}_{1}/\hbar}\left[1 + \cos\left(\frac{\delta p}{\hbar}\underbrace{\left(x - \frac{p_0}{m}t\right)}_{2}\right)\right]$$

we get two wave forms!

1 $e^{ip_0(x - \frac{p_0}{2m}t)/\hbar}$ is the carrier wave.

- if we look at a wave form, $e^{i2\pi(x-vt)/\lambda_c}$ we find that $\lambda_c = h/p_0$ is a single harmonic wave at the wavelength of the peak of the distribution, p_0.

- the speed of the carrier wave is $v_c = \frac{p_0}{2m}$ half the classical speed. (phase velocity)

2 $\cos\left(\frac{\delta p}{\hbar}\left(x - \frac{p_0}{m}t\right)\right)$ is the envelope wave.

- if we look at the waveform $\cos\left(\frac{2\pi(x - v_e t)}{\lambda_e}\right)$ we find $\lambda_e = h/\delta p$ b/c $\delta p \ll p_0$, $\lambda_e \gg \lambda_c$ the envelope has a long wavelength characterized by the width of the distribution, δp.
- the speed of the envelope is $v_e = \frac{p_0}{m}$ which is the classical speed. (group velocity)